3. EVALUATION OF TRIGONOMETRIC FUNCTIONS

In this section, we obtain values of the trigonometric functions for quadrantal angles, we introduce the idea of reference angles, and we discuss the use of a calculator to evaluate trigonometric functions of general angles.

In Definition 2.1, the domain of each trigonometric function consists of all angles \( \theta \) for which the denominator in the corresponding ratio is not zero. Because \( r > 0 \), it follows that \( \sin \theta = y/r \) and \( \cos \theta = x/r \) are defined for all angles \( \theta \). However, \( \tan \theta = y/x \) and \( \sec \theta = r/x \) are not defined when the terminal side of \( \theta \) lies along the \( y \) axis (so that \( x = 0 \)). Likewise, \( \cot \theta = x/y \) and \( \csc \theta = r/y \) are not defined when the terminal side of \( \theta \) lies along the \( x \) axis (so that \( y = 0 \)). Therefore, when you deal with a trigonometric function of a quadrantal angle, you must check to be sure that the function is actually defined for that angle.

**Example 3.1**

Find the values (if they are defined) of the six trigonometric functions for the quadrantal angle \( \theta = 90^\circ \) (or \( \theta = \pi/2 \)).

In order to use Definition 1, we begin by choosing any point \((0, y)\) with \(y > 0\), on the terminal side of the \(90^\circ\) angle (Figure 1). Because \(x = 0\), it follows that \(\tan 90^\circ\) and \(\sec 90^\circ\) are undefined. Since \(y > 0\), we have

\[
r = \sqrt{x^2 + y^2} = \sqrt{0^2 + y^2} = \sqrt{y^2} = |y| = y.
\]

Therefore,

\[
\sin 90^\circ = \frac{y}{r} = \frac{y}{y} = 1 \quad \cos 90^\circ = \frac{x}{r} = \frac{0}{y} = 0
\]

\[
\csc 90^\circ = \frac{r}{y} = \frac{y}{y} = 1 \quad \cot 90^\circ = \frac{x}{y} = \frac{0}{y} = 0.
\]

The values of the trigonometric functions for other quadrantal angles are found in a similar manner. The results appear in Table 3.1. Dashes in the table indicate that the function is undefined for that angle.

**Table 3.1**

<table>
<thead>
<tr>
<th>( \theta ) degrees</th>
<th>( \theta ) radians</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
<th>( \sec \theta )</th>
<th>( \csc \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>—</td>
<td>1</td>
<td>—</td>
</tr>
<tr>
<td>90°</td>
<td>( \pi/2 )</td>
<td>1</td>
<td>0</td>
<td>—</td>
<td>0</td>
<td>—</td>
<td>1</td>
</tr>
<tr>
<td>180°</td>
<td>( \pi )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>—</td>
<td>-1</td>
<td>—</td>
</tr>
<tr>
<td>270°</td>
<td>( 3\pi/2 )</td>
<td>-1</td>
<td>0</td>
<td>—</td>
<td>0</td>
<td>—</td>
<td>-1</td>
</tr>
<tr>
<td>360°</td>
<td>( 2\pi )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>—</td>
<td>1</td>
<td>—</td>
</tr>
</tbody>
</table>
It follows from Definition 2.1 that the values of each of the six trigonometric functions remain unchanged if the angle is replaced by a coterminal angle. If an angle exceeds one revolution or is negative, you can change it to a nonnegative coterminal angle that is less than one revolution by adding or subtracting an integer multiple of $360^\circ$ (or $2\pi$ radians). For instance,

$$\sin 450^\circ = \sin(450^\circ - 360^\circ) = \sin 90^\circ = 1.$$  

$$\sec 7\pi = \sec (7\pi - (3 \times 2\pi)) = \sec \pi = -1.$$  

$$\cos (-660^\circ) = \cos(-660^\circ + (2 \times 360^\circ)) = \cos 60^\circ = \frac{1}{2}.$$  

In Examples 3.2 and 3.3, replace each angle by a nonnegative coterminal angle that is less than one revolution and then find the values of the six trigonometric functions (if they are defined).

**Example 3.2**

$\theta = 1110^\circ$

By dividing 1110 by 360, we find that the largest integer multiple of $360^\circ$ that is less than $1110^\circ$ is $3 \times 360^\circ = 1080^\circ$. Thus, $1110^\circ - (3 \times 360^\circ) = 1110^\circ - 1080^\circ = 30^\circ$.  

(Or we could have started with $1110^\circ$ and repeatedly subtracted $360^\circ$ until we obtained $30^\circ$. ) It follows that

$$\sin 1110^\circ = \sin 30^\circ = \frac{1}{2}$$  

$$\csc 1110^\circ = \csc 30^\circ = 2$$  

$$\cos 1110^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$  

$$\sec 1110^\circ = \sec 30^\circ = \frac{2\sqrt{3}}{3}$$  

$$\tan 1110^\circ = \tan 30^\circ = \frac{\sqrt{3}}{3}$$  

$$\cot 1110^\circ = \cot 30^\circ = \sqrt{3}$$

**Example 3.3**

$\theta = -\frac{5\pi}{2}$

We repeatedly add $2\pi$ to $-\frac{5\pi}{2}$ until we obtain a nonnegative coterminal angle:

$$-\frac{5\pi}{2} + 2\pi = -\frac{\pi}{2}$$ (still negative)  

$$-\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}.$$  

Therefore, by Table 3.1 for quadrantal angles,

$$\sin \left(-\frac{5\pi}{2}\right) = \sin \left(\frac{3\pi}{2}\right) = -1$$  

$$\cot \left(-\frac{5\pi}{2}\right) = \cot \left(\frac{3\pi}{2}\right) = 0$$  

$$\cos \left(-\frac{5\pi}{2}\right) = \cos \left(\frac{3\pi}{2}\right) = 0$$  

$$\csc \left(-\frac{5\pi}{2}\right) = \csc \left(\frac{3\pi}{2}\right) = -1$$

and both $\tan \left(-\frac{5\pi}{2}\right)$ and $\sec \left(-\frac{5\pi}{2}\right)$ are undefined.
Table 3.2

<table>
<thead>
<tr>
<th>θ</th>
<th>degrees</th>
<th>θ</th>
<th>radians</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
<th>cot θ</th>
<th>sec θ</th>
<th>csc θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td></td>
<td>π/6</td>
<td></td>
<td>1/2</td>
<td>√3/2</td>
<td>√3</td>
<td>2√3/3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>45°</td>
<td></td>
<td>π/4</td>
<td></td>
<td>√2/2</td>
<td>√2/2</td>
<td>1</td>
<td>1</td>
<td>√2</td>
<td>√2</td>
</tr>
<tr>
<td>60°</td>
<td></td>
<td>π/3</td>
<td></td>
<td>√3/2</td>
<td>1/2</td>
<td>√3</td>
<td>2</td>
<td>3/3</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 3.2
Example 3.4

Find the reference angle \( \theta_R \) for each angle \( \theta \).

(a) \( \theta = 60^\circ \)  
(b) \( \theta = \frac{3\pi}{4} \)  
(c) \( \theta = 210^\circ \)  
(d) \( \theta = \frac{5\pi}{3} \).

(a) By Figure 3.2(a), \( \theta_R = \theta = 60^\circ \).

(b) By Figure 3.2(b), \( \theta_R = \pi - \theta = \pi - \frac{3\pi}{4} = \frac{\pi}{4} \).

(c) By Figure 3.2(c), \( \theta_R = \theta - 180^\circ = 210^\circ - 180^\circ = 30^\circ \).

(d) By Figure 3.2(c), \( \theta_R = 2\pi - \theta = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3} \).

The value of any trigonometric function of any angle \( \theta \) is the same as the value of the function for the reference angle, \( \theta_R \), except possibly for a change of algebraic sign.

Thus,

\[
\sin \theta = \pm \sin \theta_R, \quad \cos \theta = \pm \cos \theta_R, \]

and so forth. You can always determine the correct algebraic sign by considering the quadrant in which \( \theta \) lies.

Section 3 Problems

In problems 1 and 2, find the values (if they are defined) of the six trigonometric functions of the given quadrantal angles. (Do not use a calculator.)

1. (a) \( 0^\circ \)  
   (b) \( 180^\circ \)  
   (c) \( 270^\circ \)  
   (d) \( 360^\circ \).

   [When you have finished, compare your answers with the results in Table 3.1]

2. (a) \( 5\pi \)  
   (b) \( 6\pi \)  
   (c) \( -7\pi \)  
   (d) \( \frac{5\pi}{2} \)  
   (e) \( \frac{7\pi}{2} \).

In Problems 3 to 14, replace each angle by a nonnegative coterminal angle that is less than one revolution and then find the exact values of the six trigonometric functions (if they are defined) for the angle.

3. \( 1440^\circ \)  
4. \( 810^\circ \)  
5. \( 900^\circ \)  
6. \( -220^\circ \)  
7. \( 750^\circ \)  
8. \( 1845^\circ \)  
9. \( -675^\circ \)  
10. \( \frac{19\pi}{2} \)
11. \( 5\pi \)  
12. \( \frac{25\pi}{6} \)  
13. \( \frac{17\pi}{3} \)  
14. \( -\frac{31\pi}{4} \)
15. What happens when you try to evaluate \( \tan 900^\circ \) on a calculator? [Try it.]

16. Let \( \theta \) be a quadrant III angle in standard position and let \( \theta_r \) be its reference angle. Show that the value of any trigonometric function of \( \theta \) is the same as the value of \( \theta_r \), except possibly for a change of algebraic sign. Repeat for \( \theta \) in quadrant IV.

In problems 17 to 36, find the reference angle \( \theta_r \) for each angle \( \theta \), and then find the exact values of the six trigonometric functions of \( \theta \).

17. \( \theta = 150^\circ \)  
18. \( \theta = 120^\circ \)  
19. \( \theta = 240^\circ \)  
20. \( \theta = 225^\circ \)  
21. \( \theta = 315^\circ \)  
22. \( \theta = 675^\circ \)  
23. \( \theta = -150^\circ \)  
24. \( \theta = -5\pi/6 \)  
25. \( \theta = -60^\circ \)  
26. \( \theta = -13\pi/6 \)  
27. \( \theta = -\pi/4 \)  
28. \( \theta = 53\pi/6 \)  
29. \( \theta = -2\pi/3 \)  
30. \( \theta = 9\pi/4 \)  
31. \( \theta = 7\pi/4 \)  
32. \( \theta = -50\pi/3 \)  
33. \( \theta = 11\pi/3 \)  
34. \( \theta = -147\pi/4 \)  
35. \( \theta = -420^\circ \)  
36. \( \theta = -5370^\circ \)

37. Complete the following tables. (Do not use a calculator.)

<table>
<thead>
<tr>
<th>( \theta ) degrees</th>
<th>( \theta ) radians</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 210^\circ )</td>
<td>( 7\pi/6 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 225^\circ )</td>
<td>( 5\pi/4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 240^\circ )</td>
<td>( 4\pi/3 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 300^\circ )</td>
<td>( 5\pi/3 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 315^\circ )</td>
<td>( 7\pi/4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 330^\circ )</td>
<td>( 11\pi/6 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>θ</td>
<td>θ</td>
<td>cot θ</td>
<td>sec θ</td>
<td>csc θ</td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>210°</td>
<td>7π/6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>225°</td>
<td>5π/4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>240°</td>
<td>4π/3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300°</td>
<td>5π/3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>315°</td>
<td>7π/4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>330°</td>
<td>11π/6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

38. A calculator is set in radian mode. π is entered and the sine (SIN) key is pressed. The display shows $-4.1 \times 10^{-10}$. But we know that $\sin \pi = 0$. Explain.

In problems 59 to 62, use a calculator to verify that the equation is true for the indicated value of the angle θ.

39. $\tan \theta = \frac{\sin \theta}{\cos \theta}$ for $\theta = 35^\circ$.

40. $(\cos \theta)(\tan \theta) = \sin \theta$ for $\theta = \frac{5\pi}{7}$

41. $\cos^2 \theta + \sin^2 \theta = 1$ for $\theta = \frac{5\pi}{3}$

42. $1 + \tan^2 \theta = \sec^2 \theta$ for $\theta = 17.75^\circ$

43. Verify that for $\theta = 0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, $90^\circ$, we have $\sin \theta = \frac{\sqrt{0}}{2}$, $\frac{\sqrt{1}}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{4}}{2}$ respectively. [Although there is no theoretical significance to this pattern, people often use it as a memory aid to help recall these values of $\sin \theta$.]