The $\beta$-model for random hypergraphs

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Based on joint work with
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**Motivation**
- Higher-order interactions in networks

**The $\beta$-model for graphs**
- A linear exponential family

**$\beta$-models for hypergraphs**
- New exponential families for higher-order interactions.
  - $k$-uniform variant
  - Layered variant
  - General variant

**Parameter Estimation**

**Concluding remarks**
Modeling Higher-order Relational Data

- Standard network problem: model binary interactions between individuals.
  - Given a model, generate a graph at random from the model.
  - Given an observed network, and assuming it belongs to a particular model, estimate the parameters.
- Binary interactions $\rightarrow$ model with graphs (or digraphs).
Modeling Higher-order Relational Data

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  - Given an observed network, and assuming it belongs to a particular model, estimate the parameters.
- Binary interactions $\implies$ model with graphs (or digraphs).
- Modeling $k$-ary interactions for $k \geq 3$?
- Examples: a co-authorship data set, team structures in organizations, sets of receivers within the range of a transmitter e.t.c.
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Higher-order interactions in networks

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- Examples: a co-authorship data set, team structures in organizations, sets of receivers within the range of a transmitter e.t.c.

- One can replace each $k$-ary relation with $\binom{k}{2}$ binary relations, in other words mapping each hyperedge onto the underlying graph.

- However such a process, inevitably causes information loss.
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- Examples: a co-authorship data set, team structures in organizations, sets of receivers within the range of a transmitter e.t.c.
- One can replace each $k$-ary relation with $\binom{k}{2}$ binary relations, in other words mapping each hyperedge onto the underlying graph.
- However such a process, inevitably causes information loss.
- Natural discrete structure to model higher order interactions: hypergraphs.
A hypergraph is a pair \((V, F)\)
- \(V\): vertex set. Here: \(V = \{1, 2, \ldots, n\} = [n]\).
- \(F\): (hyper)edge set containing subsets of \(V\) of any size.

Example

Adam, Barbara and Cassandra authored a paper as a group.
Adam and David authored a second paper, and Cassandra and David authored a third paper.

\[H_1 : V = \{A, B, C, D\}, \quad F_1 = \{ABC, AD, CD\}.\]
Example

- Reducing the co-authorship data set to a graph would give $G$ the union of the two triangles $ABC$, and $ACD$.

\[ V = \{A, B, C, D\} \]
\[ E(H_1) = \{ABC, AD, CD\}. \]
\[ E(G) = \{AB, AC, AD, BC, CD\} \]

Figure: $H_1$, $G$
Example

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![Diagram](image)

$V = \{A, B, C, D\}$

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Figure: $H_1$, $G$

- Q. How many papers gave rise to $G$?
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![Diagram showing nodes A, B, C, D and edges between them]

$V = \{A, B, C, D\}$

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$E(G) = \{AB, AC, AD, BC, CD\}$

$E(H_2) = \{ABC, ACD\}$.

Figure: $H_1$, $G$, $H_2$

- Q. How many papers gave rise to $G$? 3?
Example

- Reducing the co-authorship data set to a graph would give $G$ the union of the two triangles $ABC$, and $ACD$.

![Diagram showing graphs $H_1$, $G$, and $H_2$]

$$V = \{A, B, C, D\}$$
$$E(H_1) = \{ABC, AD, CD\}.$$  
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Figure: $H_1$, $G$, $H_2$

- Q. How many papers gave rise to $G$? 3? 2?
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Reducing the co-authorship data set to a graph would give $G$ the union of the two triangles $ABC$, and $ACD$.

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So how can we model random hypergraphs?
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**Figure**: $H_1$, $G$, $H_2$

- Q. How many papers gave rise to $G$? 3? 2? 5?
- So how can we model random hypergraphs?
- Use the degree sequence;
**Motivation**

**Higher-order interactions in networks**

**Example**

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Q. How many papers gave rise to $G$? 3? 2? 5?

So how can we model random hypergraphs?

- Use the degree sequence; recall the $\beta$ model for graphs.
The $\beta$-model for graphs

A linear exponential family

$\beta$-model for graphs

Probability of edge

- $\beta_i \in \mathbb{R}$: propensity of node $i$ to have neighbors.

$$P_\beta(\text{edge } (i,j) \text{ appearing in } G) = p_{i,j} = \frac{e^{\beta_i + \beta_j}}{1 + e^{\beta_i + \beta_j}}.$$

Probability of graph

$g$: fixed graph; degree sequence $d(g) = (d_1, d_2, \ldots d_n)$;
$G$ random variable drawn from the distribution $P_\beta$:

$$P_\beta(G = g) = \exp \left\{ \sum_i \beta_i d_i - \psi(\beta) \right\},$$

$$\psi(\beta) = \sum_{i<j} \log (1 + e^{\beta_i + \beta_j}), \text{ normalizing constant.}$$
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**$\beta$-model for graphs**

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\psi(\beta) = \sum_{i<j} \log (1 + e^{\beta_i + \beta_j}), \text{ normalizing constant.}
$$

- Linear Exponential family; sufficient statistics vector is $d$. 

Despina Stasi (IIT)
β-model for graphs

- Linear Exponential family; sufficient statistics vector is \( d \).
- Well-studied model:
  - Thorough analysis in (Chatterjee et al 2011), who coined the name.
  - (Park and Newman 2004, Blitzstein and Diaconis 2009)
  - undirected version of the \( p_1 \)-model (Holland-Leinhardt 1981)
- MLE estimation via iterative proportional fitting/scalling or fixed point algorithm.
- MLE non-existence related to the polytope of degree sequences of graphs (Rinaldo et al 2012).
Degree Sequence of a Hypergraph

- $d = (d_1, d_2, \ldots, d_n)$: the degree sequence of a hypergraph.
- $d_i$: number of hyperedges of $H$ containing vertex $i$.
- Recall our co-authorship example.

Example

Figure: $H_1$

- $H_1: V = \{A, B, C, D\}, \ F_1 = \{ABC, AD, CD\}$.
- $d =$
Degree Sequence of a Hypergraph

- \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \): the degree sequence of a hypergraph.
- \( d_i \): number of hyperedges of \( H \) containing vertex \( i \).
- Recall our co-authorship example.

Example

\[
H_1 : V = \{ A, B, C, D \}, \quad F_1 = \{ ABC, AD, CD \}.
\]

\[
\mathbf{d} = (2, 1, 2, 2).
\]

Figure: \( H_1 \)
Three variants of the $\beta$-model for hypergraphs

We define three models, depending on the type of hypergraph we would like to model:

- **$k$-uniform hypergraph $\beta$-model**
  - All hyperedges in $H$ are of size $k$.
  - $k = 2$ gives the $\beta$-model for graphs.

- **Layered hypergraph $\beta$-model**
  - Allow an individual a different propensity for participating in different size group interactions.
  - Each vertex is associated with $r$ parameters $\beta_i^{(k)}$, one for integer $k$ such that size-$k$ edges are allowed in the model.

- **General hypergraph $\beta$-model**
  - One parameter $\beta_i$ for each vertex $i \in [n]$: this parameter governs the propensity of node $i$ to participate in edges of all sizes.
**$k$-uniform hypergraph $\beta$-model**

Model parameters: $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

**Probability of edge**

For a random hypergraph $H$ drawn from this distribution:

\[
P(\text{edge } (i_1i_2\ldots i_k) \text{ in } H) = p_{i_1,i_2,...,i_k} = \frac{e^{\beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_k}}}{1 + e^{\beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_k}}}
\]
$k$-uniform hypergraph $\beta$-model

Model parameters: $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

Probability of edge

For a random hypergraph $H$ drawn from this distribution:

$$P(\text{edge } (i_1, i_2, \ldots, i_k) \text{ in } H) = p_{i_1, i_2, \ldots, i_k} = \frac{e^{\beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_k}}}{1 + e^{\beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_k}}}.$$

Probability of hypergraph

$$P(H) = \exp \left\{ \sum_{i} \beta_i d_i - \psi(\beta) \right\},$$

$$\psi(\beta) = \sum_{s \in \binom{[n]}{k}} \log(1 + e^{\sum_{j \in f} \beta_j}),$$ normalizing constant.
$k$-uniform hypergraph $\beta$-model

Model parameters: $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

**Probability of edge**

For a random hypergraph $H$ drawn from this distribution:

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**Probability of hypergraph**

$$P(H) = \exp \left\{ \sum \beta_i d_i - \psi(\beta) \right\},$$

where

$$\psi(\beta) = \sum_{s \in \binom{[n]}{k}} \log(1 + e^{\sum_{s \in f} \beta_j}),$$

is the normalizing constant.
Maximum likelihood equations

- The log-likelihood function is strictly concave.
- $\hat{\beta}$: maximum likelihood estimator (MLE):
  \[
  \frac{\partial}{\partial \hat{\beta}_i} \psi(\hat{\beta}) = d_i. \]

ML Equations

The MLE $\hat{\beta}$ must satisfy the $n$ equations:

\[
d_i = \sum_{s \in \binom{[n] \setminus \{i\}}{k-1}} \frac{e^{\hat{\beta}_i + \sum_{j \in s} \hat{\beta}_j}}{1 + e^{\hat{\beta}_i + \sum_{j \in s} \hat{\beta}_j}}. \quad (1)
\]
Layered hypergraph $\beta$-model

Model parameters: for $2 \leq k \leq r$: $\beta^{(k)} = (\beta_{1}^{(k)}, \beta_{2}^{(k)}, \ldots, \beta_{n}^{(k)})$.

Given a random hypergraph $H$ drawn from this distribution:

- $P(\text{edge } (i_1 \ldots i_k) \text{ in } H) = p_{i_1, i_2, \ldots, i_k} = \frac{e^{\beta_{i_1}^{(k)} + \beta_{i_2}^{(k)} + \ldots + \beta_{i_k}^{(k)}}}{1 + e^{\beta_{i_1}^{(k)} + \beta_{i_2}^{(k)} + \ldots + \beta_{i_k}^{(k)}}}$.

- $P(H) = \exp \left\{ \sum_{k} \sum_{i} \beta_{i}^{(k)} d_{i}^{(k)} - \psi(\beta) \right\}$, with the normalizing constant
  $\psi(\beta) = \sum_{k} \sum_{f \in ([n]^{k})} \log(1 + e^{\sum_{j \in f} \beta_{j}^{(k)}})$. 
Layered hypergraph $\beta$-model

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Given a random hypergraph $H$ drawn from this distribution:

- $P(\text{edge } (i_1i_2\ldots i_k) \text{ in } H) = p_{i_1, i_2, \ldots, i_k} = \frac{e^{\beta_{i_1}^{(k)} + \beta_{i_2}^{(k)} + \ldots + \beta_{i_k}^{(k)}}}{1 + e^{\beta_{i_1}^{(k)} + \beta_{i_2}^{(k)} + \ldots + \beta_{i_k}^{(k)}}}$.

- $P(H) = \exp \left\{ \sum_k \sum_i \beta_i^{(k)} d_i^{(k)} - \psi(\beta) \right\}$, with the normalizing constant $\psi(\beta) = \sum_k \sum_{f \in (\mathbb{N})_k} \log(1 + e^{\sum_j \beta_j^{(k)}})$.

ML Equations

The MLE $\hat{\beta}$ must satisfy the $n \times (r - 1)$ equations:

$$d_i^{(k)} = \sum_{s \in \left(\mathbb{N}_{k-1}\right) \setminus \{i\}} \frac{e^{\beta_s^{(k)}}}{1 + e^{\beta_s^{(k)} + \sum_j \hat{\beta}_j^{(k)}}}$$

(2)
General hypergraph $\beta$-model

Model parameters: $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.

Given a hypergraph $H$ drawn from this distribution:

- $P(\text{edge } e = i_1i_2\ldots i_k \text{ appearing in } H) =$
  \[ p_{i_1, i_2, \ldots, i_k} = \frac{e^{\beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_k}}}{1 + e^{\beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_k}}}. \]

- $P(H) = \exp\{\sum_i \beta_i d_i - \psi(\beta)\}$, with the normalizing constant $\psi(\beta) = \sum_k \sum_{f \in \binom{[n]}{k}} \log(1 + e^{\sum_{j \in f} \beta_j}).$
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**ML Equations**

The MLE $\hat{\beta}$ must satisfy the $n$ equations:

$$d_i = \sum_k \sum_{s \in [n] \setminus \{i\}}^{k-1} \frac{e^{\hat{\beta}_i + \sum_{j \in s} \hat{\beta}_j}}{1 + e^{\hat{\beta}_i + \sum_{j \in s} \hat{\beta}_j}}$$
Fixed Point Algorithms

$k$-uniform variant

Given an observed hypergraph $H$ with degree, for $x \in \mathbb{R}^n$ define $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $i$th coordinate:

$$
\varphi_i(x) = \log d_i - \log \sum_{s \in ([n] \setminus \{i\}) \atop |s| = k-1} \left( e^{-\sum_{j \in s} x_j} + e^{x_i} \right)^{-1}
$$

- Suppose that the ML equations (1) have unique solution $\hat{\beta}$.
- Then $\hat{\beta}$ is a fixed point of the function $\varphi$. 
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$$

Suppose that the ML equations (1) have unique solution $\hat{\beta}$.
Then $\hat{\beta}$ is a fixed point of the function $\varphi$.

Algorithm

Initialize $x_0 \in \mathbb{R}^n$.

for $\ell = 1, 2, 3, \ldots$

$$
x_{\ell+1} = \varphi(x_\ell)
$$

Theorem

If the ML equations have a unique solution $\hat{\beta}$, then the sequence $\{x_\ell\}$ converges to $\hat{\beta}$. 
Fixed Point Algorithms

The fixed point algorithm, with altered $\varphi$ function, works for the other two models as well:

Layered variant

There are $r$ functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, one for each relevant edge size $k$.

$$\varphi_i^{(k)}(x) = \log d_i^{(k)} - \log \sum_{s \in \binom{[n]\setminus\{i\}}{k-1}} \left( e^{-\sum_{j \in s} x_j + e^{x_i}} \right)^{-1}$$

General Variant

For the general case, there is a single altered function $\varphi$ where the second term includes more summands.

$$\varphi_i(x) = \log d_i - \log \sum_k \sum_{s \in \binom{[n]\setminus\{i\}}{k-1}} \left( e^{-\sum_{j \in s} x_j + e^{x_i}} \right)^{-1}$$
Sanity Check

- Simulate a hypergraph $H = (V, F)$ drawn from the beta model for 3-uniform hypergraphs on 10 vertices.

$\beta = (-5.05, -0.57, 2.87, 4.85, 1.98, -6.69, -3.95, 5.97, -6.61, -4.24)$.

- Draw a bunch of hypergraphs from this model; mean sufficient statistics vector:

$\bar{d} = (6.28, 10.70, 17.59, 20.81, 16.55, 4.41, 7.47, 23.02, 4.50, 7.17)$

- Average simulated edge density = 0.33.
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- Fixed point algorithm using $\bar{d}$ as the sufficient statistic decides that MLE exists.

$$\hat{\beta} = (-4.94, -0.58, 2.81, 4.76, 1.94, -6.55, -3.86, 5.86, -6.48, -4.15).$$
**Sanity Check**

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  $$\hat{\beta} = (-4.94, -0.58, 2.81, 4.76, 1.94, -6.55, -3.86, 5.86, -6.48, -4.15).$$

- $\|\beta - \hat{\beta}\|_{\infty} = 0.14.$
## MLE existence simulations

<table>
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<td>(7, 4, 5, 4, 7, 5, 6, 1, 3, 3 )</td>
<td>No</td>
<td>80</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>(4, 2, 4, 2, 3, 4, 3, 1, 3, 4)</td>
<td>No</td>
<td>1000</td>
<td>4</td>
</tr>
</tbody>
</table>

**Table:** Convergence of Fixed Point Algorithm for 3-uniform hypergraphs on 10 vertices. Computation time: under a second.

Despina Stasi (IIT)
### MLE existence simulations

<table>
<thead>
<tr>
<th>#edges</th>
<th>Degree Sequence</th>
<th>Conv?</th>
<th>#steps</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>(152, 157, 150, 142, 150, 153, 149, 152, 150, 146, 151, 152, 151, 153, 144, 151, 157, 154, 143, 143)</td>
<td>Yes</td>
<td>74</td>
<td>n/a</td>
</tr>
<tr>
<td>900</td>
<td>(143, 130, 136, 129, 143, 144, 135, 136, 137, 130, 134, 141, 143, 133, 132, 133, 135, 132, 127, 127)</td>
<td>Yes</td>
<td>41</td>
<td>n/a</td>
</tr>
<tr>
<td>800</td>
<td>(116, 114, 127, 118, 125, 128, 123, 118, 117, 114, 126, 118, 126, 122, 109, 126, 119, 115, 113, 126)</td>
<td>Yes</td>
<td>25</td>
<td>n/a</td>
</tr>
<tr>
<td>600</td>
<td>(82, 87, 94, 95, 99, 93, 83, 78, 87, 90, 95, 81, 86, 93, 96, 98, 88, 88, 95, 92)</td>
<td>Yes</td>
<td>16</td>
<td>n/a</td>
</tr>
<tr>
<td>600</td>
<td>(88, 86, 81, 82, 87, 84, 95, 108, 100, 91, 89, 98, 86, 92, 85, 86, 90, 94, 90, 88)</td>
<td>Yes</td>
<td>19</td>
<td>n/a</td>
</tr>
<tr>
<td>500</td>
<td>(72, 79, 69, 67, 80, 77, 77, 76, 80, 79, 74, 72, 85, 70, 77, 82, 76, 65, 75, 68)</td>
<td>Yes</td>
<td>20</td>
<td>n/a</td>
</tr>
<tr>
<td>400</td>
<td>(61, 65, 69, 51, 56, 63, 58, 50, 64, 62, 57, 63, 53, 63, 61, 59, 58, 63, 65, 59)</td>
<td>Yes</td>
<td>112</td>
<td>n/a</td>
</tr>
<tr>
<td>300</td>
<td>(35, 56, 43, 47, 49, 52, 51, 42, 38, 43, 40, 41, 43, 37, 47, 55, 49, 42, 48, 42)</td>
<td>No</td>
<td>55</td>
<td>2</td>
</tr>
<tr>
<td>300</td>
<td>(50, 45, 57, 38, 48, 49, 53, 41, 44, 50, 49, 40, 48, 44, 38, 41, 39, 39, 49, 38)</td>
<td>No</td>
<td>80</td>
<td>2</td>
</tr>
<tr>
<td>300</td>
<td>(47, 45, 47, 32, 37, 52, 51, 49, 38, 44, 41, 42, 51, 42, 49, 42, 46, 48, 52)</td>
<td>No</td>
<td>56</td>
<td>2</td>
</tr>
<tr>
<td>250</td>
<td>(40, 39, 40, 39, 48, 42, 37, 36, 31, 39, 36, 35, 33, 42, 31, 42, 31, 34, 39, 36)</td>
<td>No</td>
<td>40</td>
<td>2</td>
</tr>
<tr>
<td>200</td>
<td>(34, 36, 28, 31, 29, 28, 30, 39, 21, 37, 39, 24, 29, 30, 31, 21, 36, 28, 22)</td>
<td>No</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>150</td>
<td>(25, 17, 24, 26, 19, 22, 23, 25, 17, 18, 23, 22, 19, 15, 24, 21, 26, 31, 22, 31)</td>
<td>No</td>
<td>100</td>
<td>2</td>
</tr>
<tr>
<td>100</td>
<td>(15, 14, 16, 11, 16, 15, 17, 19, 18, 15, 19, 13, 16 ,18 ,11 ,20 ,19 ,17 ,7 ,4)</td>
<td>No</td>
<td>120</td>
<td>4</td>
</tr>
<tr>
<td>90</td>
<td>(13, 15, 19, 13, 18, 20, 18, 17, 10, 10 ,13 ,16 ,11 ,7 ,7, 11,10,15,12,15)</td>
<td>No</td>
<td>120</td>
<td>4</td>
</tr>
<tr>
<td>80</td>
<td>(9,14, 14, 21, 15, 17, 18, 11, 6, 13,11,10,10,15,14,9,14,8,12,9)</td>
<td>No</td>
<td>300</td>
<td>5</td>
</tr>
<tr>
<td>70</td>
<td>(14, 11, 14, 9, 13, 15, 9, 12, 13, 12, 10, 10, 6, 11, 7, 8, 9, 11, 9, 7)</td>
<td>No</td>
<td>200</td>
<td>8</td>
</tr>
<tr>
<td>60</td>
<td>(9, 13, 9, 4, 8, 13, 8, 7, 12, 12, 12, 8, 8, 9, 9, 8, 13, 8, 9, 4)</td>
<td>No</td>
<td>3401</td>
<td>8</td>
</tr>
<tr>
<td>50</td>
<td>(9, 9, 8, 7, 5, 7, 9, 10, 5, 11, 14, 8, 6, 8, 5, 12, 5, 3, 4)</td>
<td>No</td>
<td>348</td>
<td>16</td>
</tr>
<tr>
<td>40</td>
<td>(12, 7, 7, 7, 4, 5, 9, 8, 4, 6, 5, 6, 7, 9, 5, 4, 1, 1)</td>
<td>No</td>
<td>182</td>
<td>8</td>
</tr>
<tr>
<td>30</td>
<td>(8, 4, 11, 2, 8, 6, 4, 2, 5, 4, 3, 2, 6, 5, 5, 3, 3, 3)</td>
<td>No</td>
<td>145</td>
<td>8</td>
</tr>
</tbody>
</table>

**Table:** Convergence of Fixed Point Algorithm for 3-uniform hypergraphs on 20 vertices. Computation time: under 5 secs.
### Convergence of Fixed Point Algorithm for general hypergraphs with size 2 and 3 hyperedges on 20 vertices.

<table>
<thead>
<tr>
<th># edges</th>
<th>Degree Sequence</th>
<th>Conv?</th>
<th>#steps</th>
<th>User Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1300</td>
<td>(184,185,186,187,189,186,183,189,186,185,185,188,185,186,185,187,190,184,180)</td>
<td>Yes</td>
<td>976</td>
<td>44.666</td>
</tr>
<tr>
<td>1100</td>
<td>(161,161,154,159,155,162,156,155,160,154,152,150,153,161,155,162,154,156,161,158)</td>
<td>Yes</td>
<td>40</td>
<td>1.838</td>
</tr>
<tr>
<td>900</td>
<td>(138,119,141,114,135,123,131,123,115,143,123,126,129,122,135,128,130,124,135,134)</td>
<td>Yes</td>
<td>28</td>
<td>1.229</td>
</tr>
<tr>
<td>900</td>
<td>(127,116,117,120,140,131,124,130,125,136,133,137,122,141,126,128,123,128,120,138)</td>
<td>Yes</td>
<td>26</td>
<td>1.206</td>
</tr>
<tr>
<td>800</td>
<td>(114,130,114,113,114,107,104,120,115,112,119,113,125,119,107,111,121,113,112,103)</td>
<td>Yes</td>
<td>22</td>
<td>1.058</td>
</tr>
<tr>
<td>700</td>
<td>(96,107,111,92,95,106,103,105,105,97,99,102,94,97,104,93,99,96,98,97)</td>
<td>Yes</td>
<td>16</td>
<td>0.770</td>
</tr>
<tr>
<td>600</td>
<td>(81,91,93,84,88,88,69,85,85,89,90,75,89,90,77,84,94,91,84,85)</td>
<td>Yes</td>
<td>15</td>
<td>0.717</td>
</tr>
<tr>
<td>500</td>
<td>(75,67,82,66,79,81,75,71,67,75,64,70,69,67,65,66,62,65,79,72)</td>
<td>Yes</td>
<td>10</td>
<td>0.543</td>
</tr>
<tr>
<td>400</td>
<td>(52,66,54,62,61,66,53,47,52,61,52,53,48,64,54,73,56,53,59,58)</td>
<td>Yes</td>
<td>614</td>
<td>30.431</td>
</tr>
<tr>
<td>300</td>
<td>(44,41,37,46,56,29,52,38,43,40,42,52,38,40,52,41,47,43,35)</td>
<td>No</td>
<td>50</td>
<td>2.350</td>
</tr>
<tr>
<td>300</td>
<td>(47,45,45,58,39,45,37,50,45,47,39,46,33,38,50,39,46,40,45,34)</td>
<td>No</td>
<td>100</td>
<td>5.693</td>
</tr>
<tr>
<td>200</td>
<td>(27,35,28,29,29,16,32,34,30,40,25,30,25,26,31,36,27,19,19,28)</td>
<td>No</td>
<td>51</td>
<td>2.379</td>
</tr>
<tr>
<td>100</td>
<td>(21,20,14,10,16,13,12,15,14,19,17,23,13,10,12,11,13,17, 8, 9)</td>
<td>No</td>
<td>223</td>
<td>9.625</td>
</tr>
<tr>
<td>50</td>
<td>(6,5,12,3,7,5,4,11,7,8,4,7,7,12,9,7,8,5,10, 6)</td>
<td>No</td>
<td>657</td>
<td>28.891</td>
</tr>
</tbody>
</table>

**Table:** Convergence of Fixed Point Algorithm for general hypergraphs with size 2 and 3 hyperedges on 20 vertices.
Summary and Further Discussion

- New statistical model for higher-order interactions.
- MLE estimation can be done using IPS or fixed point algorithms: the latter is fast for smaller $k$s, much slower for larger ones.
- Computations: as in the $\beta$-model for graphs, the density of the hypergraph influences existence of the MLE.
- Connections to important open problems in combinatorics:
  - Polytope of degree sequences of hypergraphs
  - Characterizations of graphical degree sequences for hypergraphs.
  - Edge-swaps (Markov bases).
Questions?

Thank you for your attention!