Between Sphere Packing and Sphere Covering

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Motivation: Arrangement of receptive fields

Arrangement of the ganglion cell receptive fields on the retinal surface:

Receptive field packing = minimal free space arrangement?
= minimal mutual information arrangement?

(Work in progress with Chris Hillar and Steve Wright)
Motivation: Chromosome organization in cell nucleus

Chromosome packing: Arrangements of overlapping ellipsoids (that are filled with little spheres) of different sizes in an ellipsoidal container.

Chromosome packing = minimal overlap arrangement?

(Joint work with Steve Wright)
Motivation: Coding theory

Talk by Richard E. Blahut at Applied Algebra Days in Madison about “how algebraic geometry failed in coding theory”.

Code words that are far apart from each other and have good decoding properties? Find sphere configurations that pack and cover well.
Sphere packing and covering: classical questions

**Sphere packing:**

- Pack $N$ identical spheres in enclosing sphere of minimal radius (with no overlap)
- Pack identical spheres as densely as possible in infinite space (with no overlap)

Quality measured by **packing density**: $\frac{\text{vol(spheres)}}{\text{vol(enclosing sphere)}} < 1$

**Sphere covering:**

- Find maximal radius of sphere such that it can be covered with $N$ identical spheres
- Cover infinite space with identical spheres as “thinnly” as possible

Quality measured by **covering density**: $\frac{\text{vol(spheres)}}{\text{vol(enclosing sphere)}} > 1$
Classical results: Infinite 2d space

Disk packing:
- Hexagonal packing has optimal density (Thue, 1910; Tóth, 1940)
- Packing density is \( \frac{\pi}{2\sqrt{3}} \approx 0.9068997 \)

Disk covering:
- Hexagonal packing has optimal density (Kershner, 1939)
- Covering density is \( \frac{2\pi}{3\sqrt{3}} \approx 1.209200 \)
Classical results: Infinite 3d space

**Sphere packing:**
- **FCC lattice** achieves optimal density (Gauss, 1831; Hales, 2005)
- Packing density is $\frac{\pi}{3\sqrt{2}} \approx 0.7404805$

**Sphere covering:**
- **BCC lattice** achieves optimal density (Bambah, 1954)
- Covering density is $\frac{25\pi}{24\sqrt{5}} \approx 1.463503$
Disk packing in enclosing disk

- Pack $N$ identical disks in an enclosing disk of minimal radius
- Consider 91 disks arranged hexagonally:

What is the best arrangement of 91 disks in an enclosing disk?

Curved Hexagonal Packing

Twisting of hexagonal arrangement allows smaller enclosing disk!

Google “circle packing Magdeburg” for best known packings up to $N = 1100$. (Only $N = 1, 2, \ldots, 13$ and $N = 19$ are proved optimal.)
Chromosome Packing: Packing spheres with overlap

- Given $N$ spheres of radius $r$, and an ellipsoidal container $\Omega$, choose centers $c_i \in \mathbb{R}^n$ so that the spheres lie within $\Omega$ and some measure of total overlap is minimized.

- Simple measure of overlap between two spheres: $2r - \|c_i - c_j\|_2$.

- Developed iterative convex optimization strategy to solve this problem (with Steve Wright)
Our algorithm recovers the curved hexagonal packings with the appropriate number of circles \( N = 19, 37, 61, 91, \text{ etc.} \) for arbitrary radius inscribed in a larger outer circle.

(a) 37 disks

(b) 37 disks, larger radii
Diagonal distortion in $\mathbb{R}^n$ (Edelsbrunner & Kerber, 2011)

**Assumptions:** Sphere centers lie on

- $\mathcal{L}$ lattice in $\mathbb{R}^d$
- $\mathcal{L}_\delta$, $\delta > 0$: 1-parameter diagonal distortion of integer grid:
  \[ e_i \mapsto e_i + \frac{\delta - 1}{d} \mathbf{1} \] and more generally \[ x \mapsto x + \frac{\delta - 1}{d} \left( \sum_{i=1}^{d} x_i \right) \mathbf{1} \]

**Examples:**

<table>
<thead>
<tr>
<th>$d$, $\delta$</th>
<th>Lattice Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall d$, $\delta = 1$</td>
<td>integer grid</td>
</tr>
<tr>
<td>$d = 2$, $\delta = \frac{1}{\sqrt{3}}, \sqrt{3}$</td>
<td>hexagonal lattice</td>
</tr>
<tr>
<td>$d = 3$, $\delta = \frac{1}{2}$</td>
<td>BCC lattice</td>
</tr>
<tr>
<td>$d = 3$, $\delta = 2$</td>
<td>FCC lattice</td>
</tr>
</tbody>
</table>
Between sphere packing and sphere covering

- \( V \): Voronoi cell of the origin
- \( \text{density}_{\mathcal{L}}(r) = \frac{\text{vol} \ B_r(0)}{\text{vol} \ V} \)
- \( \text{union}_{\mathcal{L}}(r) = \frac{\text{vol}(B_r(0) \cap V)}{\text{vol} \ V} \)
- \( \text{lin\_overlap}_{\mathcal{L}}(r) = \max \left( \frac{2r - \min_{\ell \in \mathcal{L} \setminus \{0\}} (\|\ell\|)}{2r}, 0 \right) \)
- \( \text{vol\_overlap}_{\mathcal{L}}(r) = \frac{\text{vol} \ B_r(0) - \text{vol}(B_r(0) \cap \text{vol} \ V)}{\text{vol} \ V} \)

Quality measures of a lattice:

- \( \text{Qual}_{\text{packing}}(\mathcal{L}, \omega) = \max_{r \geq 0} \{ \text{density}_{\mathcal{L}}(r) \mid \text{overlap}_{\mathcal{L}}(r) \leq \omega \} \)
- \( \text{Qual}_{\text{packing}}(\mathcal{L}, \omega) = \min_{r \geq 0} \{ \text{density}_{\mathcal{L}}(r) \mid 1 - \text{union}_{\mathcal{L}}(r) \leq \omega \} \)
- \( \text{Qual}_{\text{pack-cov}}(\mathcal{L}, \omega) = \lambda \text{Qual}_{\text{packing}}(\mathcal{L}, \omega) + (1 - \lambda) \text{Qual}_{\text{covering}}(\mathcal{L}, \omega) \)
Results

For lin_overlap:

- $\text{Qual}_{\text{packing}}(L_\delta, \omega)$ is maximized by $\delta = \sqrt{d + 1}$ independent of $\omega$: i.e. \textbf{hexagonal} lattice in dim 2, \textbf{FCC} lattice in dim 3

- $\text{Qual}_{\text{covering}}(L_\delta, \omega)$ is minimized by $\delta = 1/\sqrt{d + 1}$ independent of $\omega$ i.e. \textbf{hexagonal} lattice in dim 2, \textbf{BCC} lattice in dim 3

For vol_overlap:

- dim 2: $\text{Qual}_{\text{packing}}(L_\delta, \omega)$ is maximized by \textbf{hexagonal} lattice independent of $\omega$.

- dim 3: $\text{Qual}_{\text{covering}}(L_\delta, \omega)$ is maximized by \textbf{FCC} lattice or by \textbf{BCC} lattice depending on $\omega$. 
Results: vol\_overlap in 3d

\[ \delta = 0.5: \text{BCC lattice}, \quad \delta = 2: \text{FCC lattice} \]
References

- U., Wright: Packing ellipsoids with overlap (SIAM Review 55)

Thank you!