Smart Grid Coordination in Building HVAC Systems:
Computational Efficiency of Constrained ELOC

David I. Mendoza-Serrano and Donald J. Chmielewski
Department of Chemical and Biological Engineering
Illinois Institute of Technology.

Abstract

In the context of day-ahead or real-time electricity prices, the method of Economic Model Predictive Control (EMPC) has been shown to provide expenditure reduction in building HVAC systems, specifically if coupled with active thermal energy storage. However, due to the diurnal nature of electricity prices, these expenditure reductions can only be achieved if the EMPC prediction horizon is sufficiently large, typically greater than 12 hours. In this work we develop an alternative controller known as Constrained Economic Linear Optimal Control (ELOC). This Constrained ELOC policy is shown to yield economic performance similar to a large horizon EMPC policy, but also possesses a virtual insensitivity to horizon size. Thus, application of the Constrained ELOC policy will require a fraction of the computational effort while yielding nearly identical economic performance.

1. INTRODUCTION

In the first installment of this series, [14], it was shown that the use of active Thermal Energy Storage (TES) coupled with Economic Model Predictive Control (EMPC) could be used to
significantly reduce building HVAC operating costs if operated under day-ahead or real-time electricity price structures. It was additionally shown that forecasts of electricity price and meteorological conditions were essential to on-line implementation of EMPC and that the quality of these forecasts would significantly impact performance.

The current effort will focus on the computational aspects of implementing EMPC within a building HVAC environment. Specifically, it will be shown that the economic performance of EMPC is quite sensitive to horizon size, especially when using shorter horizons. A similar concern over sensitivity to horizon size can be found in [10]. In that work, the problem was denoted as inventory creep and was attributed to the myopic behavior resulting from the short horizon. This observation stems from the economic based optimization problem at the core of an EMPC policy, which is likely to conclude that a zeroing out of inventory (material or energy in storage) at the end of the horizon is the lowest cost solution. While such a result has little impact on a receding horizon implementation if the horizon is large, it has been shown to significantly influence performance when using short horizons, [18,19]. These notions are akin to the turnpike discussions found in [7,23], which stem from the backwards in time solution procedure suggested by the dynamic programming approach to solving optimal control problems.

The literature on EMPC, [5,22], has focused most of its attention on stability properties. Methods include the addition of terminal constraints [8,23], terminal costs [1,15] and infinite-horizon formulations [4,9,24]. More recently, the literature has turned to the question of average performance of EMPC policies [2,15,16].

A similar development of the Constrained ELOC policy can be found in [18]. However, in
that work the only disturbance considered was the price of electricity. In the building HVAC example of the current effort there will be two disturbance signals; the price of electricity and outside temperature. Thus, the current development extends the ELOC formulation to the case of two disturbances, with the second being a process disturbance.

**1.1 BUILDING HVAC CASE STUDY**

Figure 1 (left) depicts the thermal interactions between the building, the chiller and the active TES unit. The TES unit adds a degree of freedom by allowing for independent manipulation of heat flow to the chiller, $Q_c$, and heat flow from the room, $Q_r$. Energy in the TES unit, $E_s$, is the time integral of storage heat flow, $Q_s$. The electric power to drive the chiller is $P_e = \eta_e Q_c$, where $\eta_e$ is the efficiency of the chiller, assumed to be $0.3 \text{kW} / \text{kW}_e$.

Figure 1 (right) depicts the basic building configuration to be used. While the building actually contains 100 identical rooms, it is modeled as a single room to simplify computations. Each room is divided into four compartments. Compartment 0 is for air within the room, assumed to be well mixed and at temperature $T_0$. Compartment 1 is for interior walls, floors, and ceilings, all assumed to be the same concrete material. Compartment 2 is for the outside glass walls and compartment 3 is for outside air. Compartment 1 is further divided into three sub-layers, with $T_{11}$ denoting the outer layers (in contact with the room) and $T_{12}$ for the inner layer. The temperature of the window compartment, denoted by $T_{21}$, is in contact with the room on one side and the outside air on the other. The outside temperature, $T_3$ and electricity prices, $C_e$ are assumed to be disturbances. The state space model is: $\dot{s} = As + Bm + Gp$, $q = D_s s + D_m m$, where $s$ is the state vector, $m$ is the vector of manipulated variables, $p$ is the disturbance
and \( q \) indicates the point-wise-in-time restrictions: \( q_{\text{min}} \leq q(t) \leq q_{\text{max}} \), where
\[
s = [T_0, T_{11}, T_{12}, T_{21}]^T,
\]
\[
p = [T_3, C_i]^T,
\]
\[
m = [Q_x, Q_i]^T,
\]
\[
q = [T_0, E_s, Q_s, Q_i]^T,
\]
and
\[
q_{\text{max}} = [T^\text{max}_0, E^\text{max}_s, Q^\text{max}_s, Q^\text{max}_i]^T,
\]
\[
q_{\text{min}} = [T^\text{min}_0, E^\text{min}_s, Q^\text{min}_s, Q^\text{min}_i]^T.
\]
Further details on the building model can be found in [11,12,13,14].

### 1.2 REVIEW OF EMPC

Consider the process model:
\[
\dot{s} = As + Bm + Gp, \quad q = D_s s + D_m m
\]
with point-wise-in-time restrictions: \( q_{\text{min}} \leq q(t) \leq q_{\text{max}} \). To implement a predictive controller, the continuous-time model is converted to discrete-time form (with a sample-time \( \Delta t \)) and the notion of a predictive time index is introduced. The resulting model is:

\[
s_{k+i} = A_{kj} s_{kj} + B_{kj} m_{kj} + G_{kj} \hat{p}_{kj}, \quad (1)
\]
\[
q_{kj} = D_s s_{kj} + D_m m_{kj}, \quad (2)
\]
\[
q_{\text{min}} \leq q_{kj} \leq q_{\text{max}}, \quad (3)
\]
\[
s_{kj} = \hat{s}_j \quad (4)
\]

where the index \( i \) represents the actual time of the process and the index \( k \) is predictive time.

At time \( i \), the controller will be given an estimate of the initial condition, \( \hat{s}_j \), along with forecasts of the process disturbances, \( \hat{p}_{kj} \). Then, the controller must select a sequence of control inputs \( m_{kj}, k = 1 \ldots i + N - 1 \), such that the inequalities of (3) are satisfied. As there will be many sequences satisfying (3), an additional performance measure is applied via the following optimization problem:

\[
\min_{s_{kj}, m_{kj}} \left\{ \sum_{k=i}^{i+N-1} g(s_{kj}, m_{kj}, \hat{p}_{kj}) + g_N(s_{i+N|j}) \right\}
\]
\[
s.t. \quad (1), (2), (3), \text{ and } (4)
\]

If the solution to (5) is \( m^*_kj \), then the input given to the actual process,
\[ s_{i+1} = A_d s_i + B_d m_i + G_d p_i, \] is \[ m_i = m^*_i. \] At the next time step, \( i+1 \), new estimates \( \hat{s}_{i+1} \) and \( \hat{p}_{i+1} \) are calculated (based on new measurements) and given back to (5) to calculate \[ m_{i+1} = m^*_{i+1}. \]

**Example 1:** This example uses the Full Future Information (FFI) scenario of [14]. This is the case under which all future information is assumed to be known to the controller. This scenario has been chosen to exemplify the impact of horizon size on EMPC performance without having to wonder if the results obtained are due in part to forecasting issues. Outside temperature and cost of electricity will be those indicated in Figure 2. The outside temperature corresponds to historic hourly data from Houston, TX (July, 2012), [17], and the electricity prices correspond to the historic real-time prices for the same period and location, [6].

The cost of operating the plant during period \( i \) is given as: \( C_{e,i} P_{c,i} \Delta t_s \), where \( P_{c,i} \) is power to the chiller, \( C_{e,i} \) is the price of electricity and \( \Delta t_s \) is the sample-time chosen to be one hour for all examples presented in this work. Thus, an appropriate definition of the EMPC objective function is \( g_N(s_{i*N_i}) = 0 \) and

\[
g_N(s_{i*N_i}, m_{i*N_i}, p_{i*N_i}) = C_{e,i} P_{c,i} \]

(6)

The EMPC simulations to follow were implemented with Mathworks MATLAB on an Intel Pentium Dual CPU T3200 @ 2.00 GHz with 3.00 GB RAM. The simulation length was 28 days for all cases. As a baseline, the 24 hour horizon EMPC was implemented assuming zero TES, and resulted in an operating cost of $827. Assuming the TES to be 1.5 MW·hr, resulted in an operating cost (with the 24 hour EMPC) of $520 (a reduction of 37.1% compared to the baseline) and required 13 s of computational time. The same simulation with a 2 hour EMPC horizon resulted in an operating cost of $651 (a reduction of only 21.3%) and required 7 s. The
loss of economic performance is observed in Figure 3, which shows significant inventory creep for the \( N = 2 \) case.

While the computational effort of the 24 hour horizon case of Example 1 seems sufficient for on-line implementation, it is highlighted that the model is exceedingly simple with only 5 states. In addition, the model does not account for internal equipment and occupant heating, nor does it consider indoor air quality and humidity. Furthermore, if one would like to consider the larger question of system design (the subject of part 3 in this series of papers), then fast simulation times will be required to achieve computational tractability.

In the following section, a control policy alternative to EMPC will be developed. This policy will have similar economic motives, but will be a linear feedback. In section 3, this Economic Linear Optimal Control (ELOC) policy will be extended to account for point-wise-in-time constraints.

2 Economic Linear Optimal Control (ELOC)

Before we begin the ELOC development it is important to emphasize that the goal is to develop an approximation of the EMPC policy. Thus, we are free to explore other optimization problems that are similar to problem (5), but clearly different. Of course, the level of similarity will impact the quality of the approximation. As in [14] it will be convenient to distinguish the process model (representing the physical structure) from the shaping filter model (representing the disturbances). In the building HVAC case, the process model is of the following linear form

\[
\begin{align*}
    s_{i+1}^{(p)} &= A_d^{(p)} s_i^{(p)} + B_d^{(p)} m_i^{(p)} + G_d^{(p)} p_i \\
    q_i^{(p)} &= D_d^{(p)} s_i^{(p)} + D_u^{(p)} m_i^{(p)}
\end{align*}
\]
where as in Section 1, \( s^{(p)} = [T_0 \ T_{11} \ T_{12} \ T_{21} \ E_x]^T \), \( m^{(p)} = [Q_s \ Q_e]^T \), \( p^{(p)} = [T_3 \ C_e]^T \).

The average of the state, manipulated and output variables must satisfy
\[
\begin{align*}
\bar{s}^{(p)} &= A_d^{(p)} s^{(p)} + B_d^{(p)} u^{(p)} + G_d^{(p)} \bar{p} \\
\bar{q}^{(p)} &= D_x^{(p)} \bar{s}^{(p)} + D_u^{(p)} \bar{m}^{(p)}
\end{align*}
\]

(9) (10)

Deviation variables can then be defined as \( x_i^{(p)} = s_i^{(p)} - \bar{s}^{(p)} \), \( u_i^{(p)} = m_i^{(p)} - \bar{m}^{(p)} \), \( z_i^{(f)} = p_i - \bar{p} \) and \( z_i^{(p)} = q_i - \bar{q} \), which results in the following model:
\[
\begin{align*}
x_{i+1}^{(p)} &= A_d^{(p)} x_i^{(p)} + B_d^{(p)} u_i^{(p)} + G_d^{(p)} z_i^{(f)} \\
z_i^{(p)} &= D_x^{(p)} x_i^{(f)} + D_u^{(p)} u_i^{(f)}
\end{align*}
\]

(11) (12)

The disturbance sequence, \( p_i \), is assumed to be a stochastic process and the output of the following linear shaping filter.
\[
\begin{align*}
x_i^{(f)} &= A_f^{(f)} x_i^{(f)} + G_f^{(f)} w_i^{(f)} \\
z_i^{(f)} &= D_x^{(f)} x_i^{(f)}
\end{align*}
\]

(13) (14)

where \( w_i^{(f)} \) is a zero mean, white noise sequence with covariance \( \Sigma_w^{(f)} \) and \( p_i = z_i^{(f)} + \bar{p} \).

Combining the two models into a single compound system gives
\[
\begin{align*}
x_{i+1} &= A_d^{(c)} x_i + B_d^{(c)} u_i + G_d^{(c)} w_i \\
z_i &= D_x^{(c)} x_i + D_u^{(c)} u_i
\end{align*}
\]

(15) (16)

where \( x_i = \begin{bmatrix} x_i^{(p)^T} & x_i^{(f)^T} \end{bmatrix}^T \), \( u_i = u_i^{(p)} \), \( w_i = w_i^{(f)} \), \( z_i = \begin{bmatrix} z_i^{(p)^T} & z_i^{(f)^T} \end{bmatrix}^T \) and
\[
\begin{align*}
A_d^{(c)} &= \begin{bmatrix} A_d^{(p)} & D_d^{(f)} \\ 0 & A_f^{(f)} \end{bmatrix} \\
B_d^{(c)} &= \begin{bmatrix} B_d^{(p)} \\ 0 \end{bmatrix} \\
G_d^{(c)} &= \begin{bmatrix} 0 \\ G_f^{(f)} \end{bmatrix} \\
D_x^{(c)} &= \begin{bmatrix} D_x^{(p)} & 0 \\ 0 & D_x^{(f)} \end{bmatrix} \\
D_u^{(c)} &= \begin{bmatrix} D_u^{(p)} \\ 0 \end{bmatrix}
\end{align*}
\]

(17) (18)

The sequence \( p_i \) should not be confused with its associated forecasts \( \hat{p}_{k+i} = [\hat{p}_{k+i}, \hat{p}_{k+i+1}, \ldots] \).

The difference is that \( \hat{p}_{k+i} \) will be a function of the measurements and its characteristics will
be strongly influenced by the measurement structure. Consider the Zero Future Information (ZFI) scenario of [14]. In this case, the forecasts, \( \hat{P}_{kj} \), will be a function of the estimate of the filter state, \( \hat{x}^{(f)}_i \), which is a function of the current and past measurements. In the Pseudo Future Information (PFI) case of [14], similar observations can be made, although in this case the shaping filter (13)-(14) will need to be the compound PFI form discussed in [14]. In contrast, \( P_i \) is assumed to be the actual signal at time \( i \). This is distinctly different from the forecast \( \hat{P}_{kj} \). Thus, our first major modification to problem (5) is to assume that the feedback policy is a function of measurements at time \( i \). This modification is significant. As opposed to problem (5), which would appropriately be characterized as generating an Open-Loop Optimal Feedback (OLOF) policy, the optimal control problem to be proposed will produce a Closed-Loop Optimal Feedback (CLOF) policy. (Strictly speaking, the feedback implementation of EMPC makes it a function of measurements at time \( i \). However, the optimization problem it solves at each time step assumes perfect knowledge of the future even if forecasts are employed. This open-loop based solution to the optimization problem is where the distinction is made.)

Initially, assume the measurements are of a Full State Information (FSI) form, in that \( y_i = x_i \). Thus, the feedback policy at time \( i \) can be a function of \( \hat{x}_i \), which contains both the process and shaping filter state at time \( i \). Removal of this assumption, which will call for \( y_i = C x_i + \nu_i \), results in the Partial State Information (PSI) framework. In this case, it will be a simple matter to generalize from the FSI results and make the feedback policy a function of the state estimate \( \hat{x}_i \). It should be emphasized that the distinction between FSI vs PSI is unrelated to the questions of ZFI vs PFI. Specifically, the latter is dictated by the form of the shaping
filter model, while the former depends on the values assumed for \( C \) and \( \Sigma_v \). Of course, the latter will influence the dimension of \( C \) and \( \Sigma_v \).

The second major modification is to remove the inequality constraints \( q_{\text{min}}^{(p)} \leq q_i^{(p)} \leq q_{\text{max}}^{(p)} \) and replace with constraints on the statistics of the closed-loop system. Concerning the first order statistics, the mean of the output will be constrained as \( q_{\text{min}}^{(p)} \leq \tilde{q}_i^{(p)} \leq q_{\text{max}}^{(p)} \), where \( \tilde{q}_i^{(p)} \) must also satisfy (9) - (10). Concerning second order statistics, let \( \sigma_j \) denote the steady-state standard deviation of the \( j^{\text{th}} \) element of \( q_i^{(p)} \). Then we will require

\[
n_\sigma \sigma_j \leq q_{\text{max},i}^{(p)} - q_{\text{min},i}^{(p)} \quad \text{(19)}
\]

\[
n_\sigma \sigma_j \leq q_{\text{max},j}^{(p)} - q_{\text{min},j}^{(p)} \quad \text{(20)}
\]

where \( n_\sigma \) is a parameter indicating the number of standard deviations the average, \( \tilde{q}_j^{(p)} \), must be from the bounds, \( q_{\text{max},j}^{(p)} \) and \( q_{\text{min},j}^{(p)} \). The important aspect of these new constraints is that \( \sigma_j \) will be directly influenced by the feedback element. If this feedback is linear with respect to the process and shaping filter states, \( u_j = Lx_j \) (our third modification), then these steady-state standard deviations \([14]\) are calculated as

\[
\sigma_j = \sqrt{\zeta_j} \quad \text{(21)}
\]

\[
\zeta_j = \rho_j \Sigma_z \rho_j^T \quad \text{(22)}
\]

\[
\Sigma_z = (D_x^{(c)} + D_a^{(c)} L) \Sigma_x (D_x^{(c)} + D_a^{(c)} L)^T \quad \text{(23)}
\]

\[
\Sigma_x = (A_d^{(c)} + B_d^{(c)} L) \Sigma_x (A_d^{(c)} + B_d^{(c)} L)^T + G_d^{(c)} \Sigma_u G_d^{(c)} \quad \text{(24)}
\]

where \( \rho_j \) is the \( j^{\text{th}} \) row of identity with dimension equal to that of \( \Sigma_z \).

The objective function for our ELOC problem is stated as

\[
\psi = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} g(q_i^{(p)} , p_i) = \lim_{N \to \infty} E \left[ g(q_i^{(p)} , p_i) \right]
\]

In the building HVAC system, \( g(q_i^{(p)} , p_i) = P_{e_d} C_{e_d} \). Thus the objective function is
\[ \psi = \lim_{i \to \infty} E \left[ g(q_i^{(p)}, p_i) \right] = \lim_{i \to \infty} \left\{ E \left[ \tilde{P}_{e,i} \tilde{C}_{e,i} \right] + \tilde{P}_{c} \tilde{C}_c \right\} \]

where \( \tilde{P}_{e,i} = P_{e,i} - \tilde{P}_c \) and \( \tilde{C}_{e,i} = C_{e,i} - \tilde{C}_c \). Since \( \tilde{P}_{e,i} \) and \( \tilde{C}_{e,i} \) are elements of \( Z_i \) the term \( \lim_{i \to \infty} E \left[ \tilde{P}_{e,i} \tilde{C}_{e,i} \right] \) can be calculated as one of the off-diagonal elements of \( \Sigma_c \). Similarly, \( \tilde{P}_c \) is an element of \( \tilde{q}^{(p)} \) and \( \tilde{C}_c \) is an element of \( \tilde{p} \). Thus, for appropriate definitions of \( a_1, a_2 \) and \( A_0 \) the objective can be written as \( a_1^T \Sigma c a_2 + \tilde{p}^T A_0 \tilde{q}^{(p)} \) and the ELOC problem can be stated as

\[
\min_{\tilde{q}^{(p)}, \tilde{p}^{(p)}, \tilde{q}^{(p)}, \sigma_i \geq 0, \Sigma_c \geq 0} \left\{ a_1^T \Sigma c a_2 + (A_0 \tilde{p})^T \tilde{q}^{(p)} \right\}
\]

\[
s.t. \quad (9), (10), (19) - (24)
\]

Unfortunately, a computationally tractable method of solving problem (25) could not be found.

As an alternative, we propose to enforce the following additional constraint

\[ \tilde{P}_{e,i} = \alpha_1 \tilde{C}_{S,i} + \alpha_2 \tilde{C}_{D,i} + \alpha_3 \tilde{C}_{L,i} \quad (26) \]

where \( \tilde{C}_{S,i} \), \( \tilde{C}_{D,i} \) and \( \tilde{C}_{L,i} \) are all signals related to \( \tilde{C}_{e,i} \) and \( a_1, a_2, \) and \( a_3 \) are scalar parameters to be selected by the optimization. More specifically, \( \tilde{C}_{S,i} \) is the short-term average of \( \tilde{C}_{e,i} \), while \( \tilde{C}_{D,i} = \tilde{C}_{e,i} - \tilde{C}_{S,i} \). Then, \( \tilde{C}_{L,i} \) is defined as the integral of \( \tilde{C}_{D,i} \). Based on these definitions it is appropriate to assume all three are orthogonal \( E \left[ \tilde{C}_{S,i} \tilde{C}_{D,i} \right] = 0 \), \( E \left[ \tilde{C}_{S,i} \tilde{C}_{L,i} \right] = 0 \) and \( E \left[ \tilde{C}_{D,i} \tilde{C}_{L,i} \right] = 0 \). Thus, the covariance term of the objective function may be reevaluated

\[
\lim_{i \to \infty} E \left[ \tilde{P}_{e,i} \tilde{C}_{e,i} \right] = \lim_{i \to \infty} E \left[ (\alpha_1 \tilde{C}_{S,i} + \alpha_2 \tilde{C}_{D,i} + \alpha_3 \tilde{C}_{L,i}) (\tilde{C}_{S,i} + \tilde{C}_{D,i}) \right]
\]

\[
= \lim_{i \to \infty} \left\{ \alpha_1 E \left[ \tilde{C}_{S,i}^2 \right] + \alpha_2 E \left[ \tilde{C}_{D,i}^2 \right] \right\}
\]

\[
= \alpha_1 \Sigma_{C_s} + \alpha_2 \Sigma_{C_D}
\]

where \( \Sigma_{C_s} \) and \( \Sigma_{C_D} \) are the known variances of \( C_s \) and \( C_D \), respectively. Using this objective function along with constraint (26), we were able to find a computationally tractable
solution for the ELOC problem. The details of this development can be found in Appendix 1.

It is highlighted that the use of $\bar{C}_{i,j}$ in equation (26) distinguishes the current ELOC formulation from that of [18]. In essence, this term allows the resulting controller to devote sufficient effort to the attenuation of process disturbances. It is additionally noted that the use of a fourth term in equation (26) corresponding to the integral of $\bar{C}_{s,i}$ was also investigated. However it was found that this term either had no impact or in some cases degraded performance.

**Example 2**: Return to the scenario of Example 1 and replace the assumption of FFI with ZFI. Furthermore, assume the disturbances are generated by the 4th order shaping of [14]. As such, the internal states of the shaping filter will be available to the simulation and thus the FSI feedback may be implemented.

The plots of Figure 4 clearly indicate that the ELOC policy has a tendency for smart grid coordination in that heat to the chiller increases when prices are low and decreases when high. In addition, regulation of room temperature is rather aggressive in that the variance of this output is small so that the average of room temperature can be very close to its maximum of $25^\circ C$ and $\bar{P}_c$ will be minimized. This aligns with the characteristics of the EMPC trajectories as well as the objective function of the ELOC, which includes $\bar{C}_c\bar{P}_c$. Where the ELOC policy seems to fail is in the observance of constraints. However, a closer inspection indicates that it is doing exactly as instructed, in that it is observing the statistical version of the constraints. The next section will illustrate how point-wise-in-time constraints can be enforced.

**Example 3**: To implement the ELOC in the more realistic scenario of having only measurements of the disturbance and not those of the internal states of the shaping filter, the
feedback will need to be a function of the state estimates \( \hat{x}_i \), or \( u_i = L\hat{x}_i \). Appendix 2 illustrates how to extend the FSI formulation to the PSI case. Using the historic data of Figure 2 along with measurement characteristics detailed in [14], and the 4th order filter of Example 2 for state estimation, the plots of Figure 5 resulted. These plots indicate ELOC performance similar to that of Figure 4. The main difference is in the energy storage plot where a substantial deviation from the average seems to occur during days 5 and 6. This results because the 4th order shaping filter within the state estimator is having a bit of trouble estimating the short-term average of the historic data used in this example. This deviation along with all other constraint violation will be removed next.

### 3 Constrained ELOC

Let us begin by taking stock of the previous section. Basically, we have identified a feedback control policy (the ELOC policy) that is a linear function of the physical states of the process as well as the internal states of the shaping filter: \( u_i = L_{ELOC}\hat{x}_i \). Using the building HVAC example we have shown that application of the ELOC policy will result in a closed-loop trajectory that is similar to that generated by the EMPC policy.

Development of the Constrained ELOC policy begins by converting the ELOC policy from its linear feedback form to its predictive form. Once in the predictive form, one can simply impose point-wise-in-time constraints to the predicted trajectories. Specifically, the predictive form of the ELOC policy is defined as

\[
\min_{x_i, u_i, q_i} \left\{ \sum_{k=i}^{i+N-1} \phi(x_{kj}, u_{kj}) \right\} + \phi_N(x_{i+N})
\]

(27)
\[ s.t. \quad x_{k+l/k} = A_d^{(c)} x_{k+1} + B_d^{(c)} u_{k+1}, \quad k = 1, \ldots, i + N - 1 \]  
\[ x_{k+1} = \hat{x}_i \]  

Where \( A_d^{(c)} \) and \( B_d^{(c)} \) are as defined in (17), \( \phi(x, x_{i+1}) = x_{i+1}^T P_{ELOC} x_{i+1} \) and

\[
\phi(x_{k+1}, u_{k+1}) = \begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} \begin{bmatrix} Q_{ELOC} & M_{ELOC} \\ M_{ELOC}^T & R_{ELOC} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix}
\]

If one were to use the LQR inverse optimality results of [3] to generate weights \( Q_{ELOC}, R_{ELOC}, M_{ELOC} \) and \( P_{ELOC} \) that correspond to \( L_{ELOC} \), then the policy generated by (27) will be identical to \( u_i = L_{ELOC} x_i \). It is important to note that [21] guarantees that such weights can be generated for all linear feedbacks produced from a class of problems to which the ELOC problem is a member. As a convenience to the reader, the ELOC controller, \( L_{ELOC} \), of Example 3 along with its inverse optimal matrices \( Q_{ELOC}, R_{ELOC}, M_{ELOC} \) and \( P_{ELOC} \) are given in Appendix 3.

Then, the Constrained ELOC policy is simply problem (27) augmented by the following point-wise-in-time constraints

\[
z_{k+1} = D_x x_{k+1} + D_u u_{k+1}
\]

\[
\bar{q}^{(p)} - q_{\text{max}}^{(p)} \leq z_{k+1}^{(p)} \leq q_{\text{max}}^{(p)} - \bar{q}^{(p)}
\]

**Example 4:** Reconsider the scenario of Example 3. The plots of Figure 6 compare the EMPC with the Constrained ELOC (both using ZFI, PSI and a horizon of 24 hours). Clearly, the point-wise-in-time constraints are now being enforced by the Constrained ELOC and the plots indicate a qualitative similarity to the EMPC policy. Quantitatively, the Constrained ELOC policy has an expenditure of $567 over 28 days, and actually outperforms the EMPC policy with expenditures of $575.

The important feature of the Constrained ELOC policy is its virtual insensitivity with respect
to horizon size, $N$. Figure 7 compares the Constrained ELOC policy using horizons of 24 and 2 hours. Clearly, the two are remarkably similar. The expenditures using a 2 hours horizon are $552 over 28 days. The computational benefits of Constrained ELOC are also highlighted in Table 1.

The source of this horizon insensitivity stems from the fact that forecasting capabilities of the shaping filter are incorporated into the ELOC policy and through inverse optimality translated into the Constrained ELOC objective function weights, most notably $P_{ELOC}$. This final cost term represents the forecasted trajectory from the end of the horizon to infinity. Thus, a change in horizon size does not change the forecasts. It only changes point-wise-in-time constraint enforcement on the forecasted trajectory. This is in stark contrast with the EMPC policy, where the forecast after the horizon is abruptly changed to zero.

It is interesting to note that in problem (27) a portion of the predicted state, $x_{k}^{\hat{f}}$, $k = 1, \ldots, N$ can be pre-specified. This is due to the fact that the shaping filter state cannot be influenced by the manipulated variable. Thus, the only way the optimization can satisfy the shaping filter portion of (28) is to set the shaping filter states equal to those that would be generated by the forecasts. Said more compactly, $x_{k}^{(f)} = \hat{x}_{k}^{(f)}$. Thus, a reduction in computational effort can be achieved by setting the shaping filter variables equal to the forecasts, which would remove the shaping filter associated equations from (28). While this observation did improve computational efficiency for the ZFI case (reported in Table 1), it will likely be essential for the PFI case where the state of the shaping filter will be much larger.
4 CONCLUSIONS

In this work we have shown that the performance of EMPC in a building HVAC application is significantly influenced by the choice of horizon size, i.e., while short horizons will yield a computational advantage, they can also result in significant degradations in expenditure savings. A major contribution of the current effort was to show that a linear feedback policy (the ELOC policy) can be constructed to yield closed-loop trajectories very much similar to those of large horizon EMPC. This ELOC Policy is distinct from that of [18] in that it can also respond to process level disturbances and not just utility cost disturbances. Then, an extension of ELOC to the Constrained ELOC policy will yield a controller that again has characteristics similar to EMPC, but also enforces point-wise-in-time constraints. In fact, under the assumption of ZFI and a PSI structure, the Constrained ELOC outperforms the large horizon EMPC. The critical feature of this Constrained ELOC policy is its virtual insensitivity to horizon size. Thus, application of the Constrained ELOC policy with a short horizon was able to yield the best of both worlds: a reduction in computational effort while preserving (or improving) economic performance.

It is highlighted that the methodology presented can be applied to other scenarios through simple modification Equations (17) and (18). Specifically, the user would provide a building model, which in practice can be obtained through first principles modeling or through a data driven model scheme (i.e., system identification based on building response data). As for the disturbance model, the statistical parameters (mean and variance) for the electricity prices and weather conditions could be modified within the model used in this work (and presented in [14]), or again a data driven modeling scheme could be employed. It is also noted that for a
given building one may want to update the Constrained ELOC policy in response to seasonal changes in the electricity and weather characteristics. Specifically, one could re-run the ELOC problem using seasonally appropriate values for the mean and variance of these signals. Similarly, one could update the value for chiller efficiency $\eta_c$ based on a change in the average of outside temperature, which would likely result in a more appropriate ELOC policy. If one would like to make chiller efficiency a true function of outside temperature, then the linearity of the building model would be invalidated. However, as highlighted in [20], a linear dynamic model is only required to determine the Constrained ELOC objective function weights. Once these weights have been determined, the predictive model in (27) can be replaced with a nonlinear model within the on-line optimization of the Constrained ELOC policy, assuming an appropriate and sufficiently fast nonlinear solver is used for on-line implementation.

**ACKNOWLEDGMENTS**

Both authors would like to thank the National Science Foundation (CBET-0967906) for financial support.

**REFERENCES**


[20] B. P. Omell and D. J. Chmielewski, "On the Tuning of Predictive Controllers: Impact of


**APPENDIX 1**

A computationally tractable approach to solving the ELOC problem begins by redefining the shaping filter, $x^{(f)}_{i+1} = \tilde{A}^{(f)}_d x^{(f)}_i + \tilde{G}^{(f)}_d w^{(f)}_i$, $z^{(f)}_i = \tilde{D}^{(f)}_x x^{(f)}_i$, such that $z^{(f)}_i = [\tilde{C}_{S,i} \tilde{C}_{D,i} \tilde{C}_{I,i} \tilde{C}_{S,i}]$. In the 4th order shaping filter of [14], these outputs arise naturally from the constructed states. If a system identification procedure has been used to construct the shaping filter from historic data, then see [11] for an appropriate procedure.

To address the bilinearity associated with $\alpha_i \tilde{C}_{S,j}$, recognize that $\alpha_i \tilde{C}_{S,j} = \tilde{C}_{S,j}$, where $\tilde{C}_{S,j}$ is determined as the output of the following filter

$$x^{(f)}_{i+1} = \tilde{A}^{(f)}_d x^{(f)}_i + \alpha_i \tilde{G}^{(f)}_d w^{(f)}_i$$

$$\tilde{C}_{S,j} = \rho_i \tilde{D}^{(f)}_x x^{(f)}_i$$

Similar relations for the other bilinear terms can be achieved through the following compound shaping filter

$$x^{(fc)}_{i+1} = A^{(fc)}_d x^{(fc)}_i + G^{(fc)}_d w^{(fc)}_i$$  (33)
\[ z_i^{(fc)} = D_x^{(fc)} x_i^{(fc)} \]  

(34)

where

\[
x_i^{(fc)} = \begin{bmatrix} x_i^{(f_1)} & x_i^{(f_2)} & x_i^{(f_3)} & x_i^{(f_4)} \end{bmatrix}^T
\]

\[
z_i^{(fc)} = \begin{bmatrix} \tilde{C}_S,i & \tilde{C}_D,i & \tilde{C}_{I_1,i} & \tilde{T}_{3,i} \end{bmatrix}^T
\]

\[ A_d^{(fc)} = \text{diag} \left( \begin{bmatrix} A_{d_1}^{(f)} & A_{d_2}^{(f)} & A_{d_3}^{(f)} \end{bmatrix} \right) \]

\[ G_d^{(fc)} = \alpha_1 G_{d_1}^{(fc)} + \alpha_2 G_{d_2}^{(fc)} + \alpha_3 G_{d_3}^{(fc)} + G_{d_4}^{(fc)} \]

Recall that \( \rho_j \) is the \( j \)th row of identity (\( I_{4x4} \) in this case). The idea being that since each \( x_i^{(f_j)} \) signal, \( j = 1, 2, 3 \), is driven by the same white noise sequence, \( w_i^{(f_j)} \), each will be identical to \( x_i^{(f_j)} \), with the exception of being scaled with respect to \( \alpha_j \). Then, the process and shaping filter model can be redefined as

\[
x_{i+1} = A_d^{(e)} x_i + B_d^{(e)} u_i + G_d^{(e)} w_i
\]

(35)

\[
z_i = D_x^{(e)} x_i + D_u^{(e)} u_i
\]

(36)

where

\[
x_i = \begin{bmatrix} x_i^{(p)} & x_i^{(fc)} \end{bmatrix}^T, \quad z_i = \begin{bmatrix} z_i^{(p)} & z_i^{(0)} \end{bmatrix}^T, \quad w_i = w_i^{(f_j)}, \quad u_i = u_i^{(p)}
\]

\[
A_d^{(e)} = \begin{bmatrix} A_d^{(p)} & G_d^{(p)} \Pi D_x^{(fc)} \\ 0 & A_d^{(fc)} \end{bmatrix}, \quad B_d^{(e)} = \begin{bmatrix} B_d^{(p)} \\ 0 \end{bmatrix}, \quad G_d^{(e)} = \begin{bmatrix} 0 \\ G_d^{(fc)} \end{bmatrix}, \quad \rho_j = \rho_j D_x^{(e)}
\]

\[
D_x^{(e)} = \begin{bmatrix} D_x^{(p)} \\ 0 \end{bmatrix}, \quad D_u^{(e)} = \begin{bmatrix} D_u^{(p)} \end{bmatrix}
\]

\[
\text{Diag}(\rho_j) = \begin{bmatrix} \rho_1 D_x^{(f_1)} & 0 & 0 & 0 \\ 0 & \rho_2 D_x^{(f_2)} & 0 & 0 \\ 0 & 0 & \rho_3 D_x^{(f_3)} & 0 \\ 0 & 0 & 0 & \rho_4 D_x^{(f_4)} \end{bmatrix}
\]

19
\[ \Pi = \begin{bmatrix} \rho_3 \\ \rho_1 + \rho_2 \end{bmatrix}, \quad \pi = \begin{bmatrix} 0 & \eta_c \end{bmatrix} \] and \( z_i^{(0)} \) is defined as \( \tilde{P}_{C,i} - \tilde{C}_{S,i} - \tilde{C}_{D,i} - \tilde{C}_{I,i} \). Using (35) - (36) the ELOC problem can be defined as

\[
\psi = \min_{\sigma_j > 0, \beta_j > 0, \Sigma_i \geq 0} \alpha_i \Sigma_i \geq \alpha_2 \Sigma_i + (A_0 \bar{p})^T \bar{q}(p) \quad (37)
\]

subject to

\[
\bar{s}(p) = A_d(p) \bar{s}(p) + B_d(p) \bar{m}(p) + G_d(p) \bar{p}
\]

\[
\bar{q}(p) = D_s(p) \bar{s}(p) + D_u(p) \bar{m}(p)
\]

\[
n_{q,j} \sigma_j \leq \bar{q}_{j}(p) - d_{\min,j} \quad j = 1, \ldots, n_q
\]

\[
n_{q,j} \sigma_j \leq q_{\max,j} - \bar{q}_{j}(p) \quad j = 1, \ldots, n_q
\]

\[
\zeta_j < \sigma_j^2 \quad j = 1, \ldots, n_q
\]

\[
\Sigma_i = (A_d^{(c)} + B_d^{(c)} L) \Sigma_i (A_d^{(c)} + B_d^{(c)} L)^T + G_d^{(c)} \Sigma_i G_d^{(c)^T}
\]

\[
\zeta_j = \rho_j (D_{d}^{(c)} + D_{u}^{(c)} L) \Sigma_i (D_{d}^{(c)} + D_{u}^{(c)} L)^T \rho_j^T \quad j = 1, \ldots, n_q + 1
\]

\[
\zeta_{n_q+1} < \varepsilon
\]

The last constraint \( \zeta_{n_q+1} < \varepsilon \) is enforcing the condition \( \lim_{\varepsilon \rightarrow \infty} E[(z_i^{(0)})^2] < \varepsilon \), and is essentially requiring \( \tilde{P}_{C,i} \approx \alpha_1 \tilde{C}_{S,i} + \alpha_2 \tilde{C}_{D,i} + \alpha_3 \tilde{C}_{I,i} \). The only computational challenges associated with problem (37) concern the reverse convexity of (42) and the non-linearity of (43) - (44). However, the following Theorem (a simple extension of that found in [3]) will exactly convert (43) - (44) to convex constraints.

**Theorem:** There exists \( \Sigma_i \geq 0 \), stabilizing \( L \), and \( \zeta_j \geq 0 \), \( j = 1, \ldots, n_q + 1 \) such that

\[
\Sigma_i = (A_d^{(c)} + B_d^{(c)} L) \Sigma_i (A_d^{(c)} + B_d^{(c)} L)^T + G_d^{(c)} \Sigma_i G_d^{(c)^T}
\]

\[
\zeta_j = \rho_j (D_{d}^{(c)} + D_{u}^{(c)} L) \Sigma_i (D_{d}^{(c)} + D_{u}^{(c)} L)^T \rho_j^T \quad j = 1, \ldots, n_q + 1
\]

\[
\zeta_j < \zeta_j^2 \quad j = 1, \ldots, n_q + 1
\]
if and only if there exists \( X > 0 \), \( Y \), and \( \zeta_j > 0 \), \( j = 1, \ldots, n_q + 1 \) such that

\[
\begin{bmatrix}
X & (A_d^{(e)}X + B_d^{(e)}Y) & G_d^{(e)} \\
(A_d^{(e)}X + B_d^{(e)}Y)^T & X & 0 \\
G_d^{(e)^T} & 0 & \Sigma_w^{-1}
\end{bmatrix} > 0
\] (49)

\[
\begin{bmatrix}
\zeta_j & \rho_j (D_x^{(e)}X + D_u^{(e)}Y) \\
(D_x^{(e)}X + D_u^{(e)}Y)^T & \rho_j^T & X
\end{bmatrix} > 0, \ j = 1, \ldots, n_q + 1
\] (50)

and \( \zeta_j < \overline{\zeta_j}^2 \), \( j = 1, \ldots, n_q + 1 \) (51)

Using Theorem A1, the ELOC problem can be re-stated as:

\[
\psi = \min_{\pi^{(p)}, \sigma^{(p)}, \alpha_1, \alpha_2, \alpha_3, \alpha_s, \alpha_{s+1, s+2, \ldots, s+5}} \left\{ \alpha_1\Sigma_{c_s} + \alpha_2\Sigma_{c_0} + (A_b\bar{p})^T \bar{Q}^{(p)} \right\}
\] (52)

\[
s.t. \quad (38) - (42), (45), (49), (50)
\]

It should be emphasized that the linearity of \( G_d^{(e)} \) with respect to \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) guarantees that constraint (49) is a Linear Matrix Inequality (LMI). Finally, the non-convex constraints of (42), are addressed using the branch and bound procedure detailed in [21]. Use of this branch and bound algorithm to solve (52) guarantees that the found solution is the globally optimal solution.

The solution to problem (52) will generate a set of optimal matrices \( X^* > 0 \) and \( Y^* \). From these, a feasible feedback is calculated as \( L_{(e)}^* = Y^* (X^*)^{-1} \). Due to the expansion of the shaping filter model (to account for the \( \alpha_j \) parameters) this feedback gain will be of the following form

\[
L_{(e)}^* = \begin{bmatrix} L_{(e)} \ L_{(f_1)} \ L_{(f_2)} \ L_{(f_3)} \ L_{(f_4)} \end{bmatrix}
\] (53)

since problem (52) believes this gain will be a linear feedback of the state \( [\chi_{(e)}^T \ \chi_{(p)}^T \ ^T] \). To
be an appropriate feedback for the original system, equations (15) - (16), the gain should be computed as

\[ L_{ELOC} = [L^*_{(p)}, L^*_{(f)}] \quad \text{where} \quad L^*_{(f)} = \sum_{j=1}^{3} L^*_{(f)} \alpha_j^* + L^*_{(fj)} \]  

(54)

And the \( \alpha_j^* \)s are the optimal values from problem (52).

**APPENDIX 2**

In the PSI case, the compound system analogous to (15) – (16) is

\[ x_{i+1} = A_d^{(c)} x_i + B_d^{(c)} u_i + G_d^{(c)} w_i \]

(55)

\[ z_i = D_d^{(c)} x_i + D_u^{(c)} u_i \]

(56)

\[ y_i = C^{(c)} x_i + v_i \]

(57)

where \( x_i = [x_i^{(p)}^T \quad z_i^{(f)}]^T \), \( u_i = u_i^{(p)} \), \( w_i = [w_i^{(p)} \quad w_i^{(f)}]^T \), \( z_i = [z_i^{(p)}^T \quad z_i^{(f)}]^T \).

\[ A_d^{(c)} = \begin{bmatrix} A_d^{(p)} & G_d^{(p)} D_d^{(f)} \\ 0 & A_d^{(f)} \end{bmatrix} \quad B_d^{(c)} = \begin{bmatrix} B_d^{(p)} \\ 0 \end{bmatrix} \quad G_d^{(c)} = \begin{bmatrix} G_d^{(p2)} & 0 \\ 0 & G_d^{(f)} \end{bmatrix} \]

\[ D_d^{(c)} = \begin{bmatrix} D_d^{(p)} & 0 \\ 0 & D_d^{(f)} \end{bmatrix} \quad D_u^{(c)} = \begin{bmatrix} D_u^{(p)} \\ 0 \end{bmatrix} \quad C^{(c)} = \begin{bmatrix} C^{(p)} & 0 \\ 0 & C^{(f)} \end{bmatrix} \]

\[ \Sigma_w^{(c)} = \begin{bmatrix} \Sigma_w^{(p)} & 0 \\ 0 & \Sigma_w^{(f)} \end{bmatrix} \quad \Sigma_v^{(c)} = \begin{bmatrix} \Sigma_v^{(p)} & 0 \\ 0 & \Sigma_v^{(f)} \end{bmatrix} \]

\[ G_d^{(p2)} = \Delta t_d G_d^{(p2)} \quad G_d^{(p)} = [0 \quad 0 \quad 0 \quad 1]^T \quad \Sigma_w^{(p)} = S_w^{(p)} / \Delta t_d \quad S_w^{(p)} = 1 \quad C^{(f)} = D_d^{(f)} \]

\[ C^{(p)} = \begin{bmatrix} \rho_i^T & \rho_i^T \end{bmatrix} \quad \Sigma_v^{(p)} = I / \Delta t_v \quad \Sigma_v^{(f)} = 3^2 I / \Delta t_v. \]

Using (59) and (57) the following optimal state estimator can be constructed

\[ \hat{x}_{i+1} = A_d^{(c)} \hat{x}_i + B_d^{(c)} u_i + K^{(c)} \eta_{i+1} \]

(58)

where \( \eta_{i+1} = y_{i+1} - C^{(c)} (A_d^{(c)} \hat{x}_i + B_d^{(c)} u_i) \), \( K^{(c)} = H C^{(c)^T} \left(C^{(c)} H C^{(c)^T} + \Sigma_v^{(c)}\right)^{-1} \), where \( H \) is the positive definite solution to
\[ H = A_d(c) \left[ H - HC(c)^\top \left( C(c)HC(c)^\top + \Sigma_v(c) \right)^{-1} C(c)H \right] A_d(c)^\top + G_d(c)\Sigma_u G_d(c)^\top \]

A convenient feature of the optimal state estimator is that the innovation sequence, \( \eta_i \), is white noise with a covariance \( \Sigma_{\eta}(c) = C(c)HC(c)^\top + \Sigma_v(c) \). Thus, equation (58) can be used as the starting point for the convex formulation of the PSI version of ELOC (just as (15) was the starting point for the FSI case). The point to recognize is that the Kalman gain, \( K(c) \), should be partitioned into two blocks: process rows and shaping filter rows

\[
K(c) = \begin{bmatrix} K(p) \\ K(f) \end{bmatrix}
\]

Then, the PSI process, analogous to (35) - (36), is defined as

\[
\hat{x}_{i+1} = A_d(c)\hat{x}_i + B_d(c)u_i + K(c)\eta_{i+1}
\]

\[
z_i = D_x(c)\hat{x}_i + D_u(c)u_i + D_x^\top x_i^f
\]

where \( A_d(c) \), \( B_d(c) \), \( D_x(c) \) and \( D_u(c) \) are unchanged and

\[
K(c) = \begin{bmatrix} K^{(p)} \\ K^{(f)} \end{bmatrix}
\]

\[
K^{(f)} = \alpha_1K^{(f)} + \alpha_2K^{(f)} + \alpha_3K^{(f)} + K^{(f)}
\]

\[
K^{(f)} = \begin{bmatrix} K^{(f)} \ 0 \ 0 \\ 0 \ K^{(f)} \ 0 \\ 0 \ 0 \ K^{(f)} \end{bmatrix}
\]

\[
K^{(f)} = \begin{bmatrix} 0 \ 0 \ 0 \ K^{(f)} \end{bmatrix}
\]

\[
D_x(c) = \begin{bmatrix} D_x(c) \ 0 \\ 0 \ 0 \end{bmatrix}
\]

\[
\tilde{x}_i(p) = x_i(p) - \hat{x}_i(p)
\]

and has a covariance of \( \Sigma_{\hat{x}}^{(p)} \) from

\[
\Sigma_{\hat{x}}^{(c)} = \begin{bmatrix} \Sigma_{\hat{x}}^{(c)} \ \Sigma_{\hat{x}}^{(f)} \\ \Sigma_{\hat{x}}^{(f)} \ \Sigma_{\hat{x}}^{(f)} \end{bmatrix} = HC(c)^\top \left( C(c)HC(c)^\top + \Sigma_v(c) \right)^{-1} C(c)H
\]

Then, if \( u_i = \hat{L}_i \) and the fact that \( E[\hat{x}_i\tilde{x}_i^\top] = 0 \) is employed, one finds the following
covariance relations

\[
\Sigma_z = \left( A_d^{(e)} + B_d^{(e)} L \right) \Sigma_z \left( A_d^{(e)} + B_d^{(e)} L \right)^T + K^{(e)} \Sigma_q K^{(e)^T}
\]

\[
\zeta_j = \rho_j \left[ \left( D_z^{(e)} + D_u^{(e)} L \right) \Sigma_z \left( D_z^{(e)} + D_u^{(e)} L \right)^T + D_y^{(p)} \Sigma_q D_y^{(p)^T} \right] \rho_j^T
\]

Re-application of Theorem A1 results in the following LMI constraints

\[
\begin{bmatrix}
X & \left( A_d^{(e)} X + B_d^{(e)} Y \right) & K^{(e)} \\
\left( A_d^{(e)} X + B_d^{(e)} Y \right)^T & X & 0 \\
K^{(e)^T} & 0 & \left( \Sigma_q^{(e)} \right)^{-1}
\end{bmatrix} > 0
\] (61)

\[
\begin{bmatrix}
\zeta_j - \rho_j D_y^{(p)} \Sigma_q D_y^{(p)^T} \rho_j^T & \rho_j \left( D_z^{(e)} X + D_u^{(e)} Y \right) \\
\left( D_z^{(e)} X + D_u^{(e)} Y \right)^T \rho_j^T & X
\end{bmatrix} > 0, \quad j = 1, \ldots, n_q + 1
\] (62)

Re-evaluation of the objective function with

\[
\tilde{p}_{c,j} = \alpha_1 \hat{C}_{s,j} + \alpha_2 \hat{C}_{d,j} + \alpha_3 \hat{C}_{i,j}
\] (63)

yields

\[
\lim_{i \rightarrow \infty} E \left[ \tilde{p}_{c,j} \hat{C}_{c,j} \right] = \lim_{i \rightarrow \infty} \left\{ \alpha_1 \left( E \left[ \hat{C}_{s,j} \right] + E \left[ \hat{C}_{s,j} \hat{C}_{d,j} \right] \right) + \alpha_2 \left( E \left[ \hat{C}_{d,j} \hat{C}_{s,j} \right] + E \left[ \hat{C}_{d,j} \right] \right) + \alpha_3 \left( E \left[ \hat{C}_{i,j} \hat{C}_{s,j} \right] + E \left[ \hat{C}_{i,j} \hat{C}_{d,j} \right] \right) \right\} = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3
\]

with

\[
d_1 = \rho_1 \Sigma_z \rho_1^T + \rho_2 \Sigma_z \rho_2^T , \quad d_2 = \rho_2 \Sigma_z \rho_2^T + \rho_3 \Sigma_z \rho_3^T , \quad d_3 = \rho_3 \Sigma_z \rho_3^T + \rho_4 \Sigma_x \rho_4^T
\]

\[
\Sigma_x = \tilde{A}_d^{(f)} \Sigma_x \tilde{A}_d^{(f)^T} + \tilde{G}_d^{(f)} \Sigma_x \tilde{G}_y^{(f)^T}, \quad \Sigma_z = \tilde{C}_x \Sigma_x \tilde{C}_x^T, \quad \Sigma_z = \Sigma_x^T
\]

Finally, the PSI version of the ELOC problem is stated as
\[ \psi = \min_{\tau^{(p)}, \eta^{(p)}, \bar{q}^{(p)}, \alpha_1, \alpha_2, \alpha_3} \{ \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + (A_j \bar{p})^T \bar{q}^{(p)} \} \]
\[ \text{s.t.} \quad (38) - (42), (45), (61), (62) \]

**APPENDIX 3**

For Example 3, \( L_{(\infty)} = \begin{bmatrix} L_{(p)}^* & L_{(f_1)}^* & L_{(f_2)}^* & L_{(f_3)}^* \end{bmatrix} \) where

\[
L_{(p)}^* = \begin{bmatrix} 0 & 0.0023 & 0.0012 & 0 & -0.0003 \\ 0 & 0 & 0 & 0 & 0.0001 \end{bmatrix}
\]

\[
L_{(f_1)}^* = \begin{bmatrix} 0 & 0 & -3.2486 & 0 & 0 \\ 0 & 0 & 3.3089 & 0 & 0 \end{bmatrix}
\]

\[
L_{(f_2)}^* = \begin{bmatrix} 0.0130 & -0.0009 & 0 & 0 & 0 \\ -0.0037 & 0.0003 & 0 & 0 & 0 \end{bmatrix}
\]

\[
L_{(f_3)}^* = \begin{bmatrix} 0.3350 & -0.0230 & 0 & -3.2452 & 0 \\ -0.0968 & 0.0066 & 0 & 3.3079 & 0 \end{bmatrix}
\]

\[
L_{(f_4)}^* = \begin{bmatrix} 1.3665 & -0.0936 & 20.2018 & -14.2702 & 0 \\ -0.3947 & 0.0270 & 0.1468 & -0.1080 & 0 \end{bmatrix}
\]

and \( \alpha_1^* = 7.8463 MW^2/hr/\$, \( \alpha_2^* = -4.6828 MW^2/hr/\$ \) and \( \alpha_3^* = -8.7867 MW^2/hr/\$ \).

Thus

\[
L_{ELOC} = \begin{bmatrix} 0 & 0.0023 & 0.0012 & 0 & -0.0003 & 30.1635 & -1.6378 & 0.1123 & -5.2878 & 14.2443 \\ 0 & 0 & 0 & 0 & 0.0001 & -15.5833 & 0.4731 & -0.0324 & 26.1091 & -29.1734 \end{bmatrix}
\]

(65)

*Theorem A2 (from [3]):* If there exists \( P > 0 \) and \( R \) such that

\[
\begin{bmatrix}
P - A_d^T PA_d + L \tau^T (R + B_d^T PB_d) L & \ast \\
-(R + B_d^T PB_d) L - B_d^T PA_d & R
\end{bmatrix} > 0
\]

then \( Q \triangleq P - A_d^T PA_d + L \tau^T (R + B_d^T PB_d) L \) and \( M \triangleq -(R + B_d^T PB_d) L - B_d^T PA_d \) will be such that
\[
\begin{bmatrix}
Q & M \\
M^T & R
\end{bmatrix} > 0
\]

And \( P \) and \( L \) will satisfy

\[
0 = A_d^T PA_d + Q - (M + A_d^T PB_d)(R + B_d^T PB_d)^{-1}(M + A_d^T PB_d)^T,
\]

\[
L = -(R + B_d^T PB_d)^{-1}(M + A_d^T PB_d)^T.
\]

Using Theorem A2, the following weights can be generated and will produce the predictive form of the ELOC policy of (65)

\[
R_{ELOC} = \begin{bmatrix}
11.8961 & 7.2991 \\
7.2991 & 8.0397
\end{bmatrix}
\]

\[
P_{ELOC} = \begin{bmatrix}
P_{ELOC}^{11} & P_{ELOC}^{12} \\
P_{ELOC}^{21} & P_{ELOC}^{22}
\end{bmatrix}
\]

\[
P_{ELOC}^{11} = \begin{bmatrix}
3.2565 & -68.7075 & 65.9624 & -0.3132 & -0.0047 \\
-68.7075 & 1548.3539 & -1486.6413 & 7.1538 & 0.0008 \\
65.9624 & -1486.6413 & 1427.4137 & -6.8665 & -0.0013 \\
-0.3132 & 7.1538 & -6.8665 & 0.0338 & -0.0002 \\
-0.0047 & 0.0008 & -0.0013 & -0.0002 & 0.0002
\end{bmatrix}
\]

\[
P_{ELOC}^{12} = \begin{bmatrix}
-9.8391 & -20.2813 & 2.2380 & -6.9726 & -0.2504 \\
3.0372 & 1.3234 & 0.5126 & 3.2002 & -1.1395 \\
-2.5596 & -7.5819 & -0.4312 & -1.7037 & -0.0803 \\
-0.9219 & -0.7353 & 0.0520 & -0.8986 & 0.5106 \\
0.2495 & 1.0394 & -0.0820 & 0.1111 & 0.1262
\end{bmatrix}
\]

\[
P_{ELOC}^{22} = \begin{bmatrix}
2624.0892 & 1198.4331 & -101.7759 & 1652.0396 & -848.8626 \\
1198.4331 & 26534.3315 & -1618.2108 & -487.6575 & 3109.4071 \\
1652.0396 & -487.6575 & -37.4149 & 13902.7978 & -1270.7266 \\
-848.8626 & 3109.4071 & -173.5993 & -1270.7266 & 1512.2086
\end{bmatrix}
\]

where \( P_{ELOC}^{21} = P_{ELOC}^{12} \cdot Q_{ELOC} \) and \( M_{ELOC} \) result from the algebraic relations of Theorem A2. It should be noted that these weights are not unique and will likely vary depending on
the type and version of LMI solver used. The important point is that use of $Q_{ELOC}$, $R_{ELOC}$, $M_{ELOC}$ within the LQR problem will generate $P_{ELOC}$ and $L_{ELOC}$. 
Table 1: Comparison of EMPC and Constrained ELOC with 1.5 $MW_{hr}$ of TES capacity for 28 days simulation historic data of Figure 2.

<table>
<thead>
<tr>
<th>Controller</th>
<th>Expenditure</th>
<th>Percent Reduction</th>
<th>Computational Effort (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EMPC (FFI) with No TES</td>
<td>$827</td>
<td>---</td>
<td>13</td>
</tr>
<tr>
<td>EMPC (FFI) $N = 24$ hrs</td>
<td>$520</td>
<td>37.1%</td>
<td>13</td>
</tr>
<tr>
<td>EMPC (ZFI PSI) $N = 24$ hrs</td>
<td>$575</td>
<td>30.5%</td>
<td>13</td>
</tr>
<tr>
<td>EMPC (ZFI PSI) $N = 2$ hrs</td>
<td>$673</td>
<td>18.6%</td>
<td>7</td>
</tr>
<tr>
<td>Constrained ELOC (ZFI PSI) $N = 24$ hrs</td>
<td>$567</td>
<td>31.4%</td>
<td>13</td>
</tr>
<tr>
<td>Constrained ELOC (ZFI PSI) $N = 2$ hrs</td>
<td>$552</td>
<td>33.3%</td>
<td>7</td>
</tr>
</tbody>
</table>
Figure 1: Left: Process diagram for HVAC system with TES. Right: Description of building zones.
Figure 2: Historic data used in Examples 1, 3 and 4 (Houston, TX, July 2012).
Figure 3: Example 1 comparison of EMPC with a 24 hour horizon (solid line) and EMPC with a 2 hour horizon. Both are FFI.
Figure 4: Example 2 comparison of FSI ELOC and FSI EMPC (24 hour horizon). Both are ZFI and use a 4th order forecasting model.
Figure 5: Example 3 comparison of PSI ELOC and PSI EMPC (24 hour horizon). Both are ZFI and use a 4th order forecasting model.
Figure 6: Example 4 comparison of Constrained ELOC and EMPC. Both are PSI/ZFI and with horizons of 24 hours.
Figure 7: Example 4 comparison of horizon size for Constrained ELOC with PSI/ZFI. Solid - 24 hour horizon. Dashed - 2 hour horizon.