**Value-Optimal Sensor Network Design Using the Generalized Benders Decomposition**

<table>
<thead>
<tr>
<th>Journal:</th>
<th><em>Industrial &amp; Engineering Chemistry Research</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript ID</td>
<td>ie-2017-01865t</td>
</tr>
<tr>
<td>Manuscript Type</td>
<td>Article</td>
</tr>
<tr>
<td>Date Submitted by the Author:</td>
<td>03-May-2017</td>
</tr>
</tbody>
</table>
| Complete List of Authors: | Zhang, Jin; Illinois Institute of Technology, Department of Chemical and Biological Engineering  
Chmielewski, Donald; Illinois Institute of Technology, Department of Chemical and Biological Engineering |
Value-Optimal Sensor Network Design Using the Generalized Benders Decomposition

Jin Zhang, Donald J. Chmielewski*

Department of Chemical & Biological Engineering, Illinois Institute of Technology, Chicago, IL 60616

Abstract

The problem of value-optimal sensor network design for linear systems has been shown to be of the nonconvex mixed integer programming class. While the branch and bound search procedure can be used to obtain a global solution, such a method is limited to fairly small systems. The bottleneck is that during each iteration of the branch and bound search, a fairly slow Semi-Definite Programming (SDP) problem must be solved to its global optimum. In this paper, it is demonstrated that an equivalent reformulation of the nonconvex mixed integer programming problem and subsequent application the Generalized Benders Decomposition (GBD) algorithm will result in massive reductions in computational effort. While the proposed algorithm has to solve multiple mixed integer linear programs, this increase in computational effort is significantly outweighed by a reduction in the number of SDP problems that must be solved.

Keywords: Sensor Placement; Optimization; Generalized Benders Decomposition; Linear Matrix Inequalities

1. Introduction

In the seminal work of Bagakewicz, [1], the notion of cost optimal Sensor Network Design (SND) was introduced. The cost optimal framework was simply stated as: minimize the capital cost of sensors subject to constraints on the performance of the resulting estimator.
One class of performance criteria was a set of bounds limiting the allowable estimation error variance. The intuitive trade-off was that more/higher quality sensors would reduce error variances, but do so at a higher capital cost. The solution method advocated in Bagajewicz [1], was a tree-search procedure. In Chmielewski et. al. [2], the cost optimal SND problem was reformulated using the method of SemiDefinite Programing (SDP) to arrive at a Mixed Integer Convex Program (MICP) that could be solved using the branch and bound algorithm. In addition, the work extended the steady-state framework of [1] to the case of dynamic open-loop processes. To address closed-loop processes, Chmielewski & Peng, [3], and Peng & Chmielewski, [4], created a similar formulation, but included the feedback element as a design variable. This pair of papers also extended the design question to include actuator selection, so that both hardware elements (sensors and actuators) could be selected simultaneously.

In a second seminal paper from the Bagajewicz group, [5], the notion of value (or profit) based SND was introduced. In that work, it was postulated that an improved sensor network would enhance the economics of the process by impacting its operating cost. Thus, the value-optimal SND problem is similar to the cost-optimal version, but augments the capital cost objective function with an operating cost term to arrive at a Net Present Value (NPV) based objective function. In Nguyen and Bagajewicz, [6], a cut-set based tree search algorithm is advocated to solve the value-optimal SND problem. In Peng & Chmielewski, [7], the value-optimal SND formulation is extended to closed-loop dynamic systems by combining SND notions with the minimally backed-off operating point notions of Economic Linear Optimal Control (ELOC). For additional discussion of ELOC, see Peng et.al., [8], and Omell and Chmielewski, [9]. Since the ELOC approach is also grounded in SDP, the resulting MICP could also be solved using branch and bound.

In 2015, a new solution method for the ELOC problem was developed, [10]. In this work, the previous branch and bound procedure was replaced with an application of the Generalized Benders Decomposition (GBD), and resulted in orders of magnitude reductions
in computational effort. In 2016, a similar, though distinct, application of the GBD to the
cost optimal SND problem also yielded orders of magnitude reductions in computational
effort, [11]. The current paper aims to apply the GBD to the value-optimal SND problem
for both steady-state systems and closed-loop dynamic processes.

The remainder of this section will review both the cost-optimal and value-optimal SND
problems, using a steady-state perspective. In addition, the essential elements of the GBD
algorithm will be reviewed. In Section II, a re-statement of the original value-optimal SND
problem will be shown to be amiable to the GBD algorithm. However, it will also be shown
that the resulting relaxed master problem will contain non-convex constraints beyond the
expected integer variables. Thus, two theorems will be provided that will ultimately yield
a relaxed master problem in the form of a Mixed Integer Linear Program (MILP) from
which solutions can be obtained with extreme efficiency. Two examples are then provided
to illustrate the computational advantage of the GBD approach. The final section will
investigate application of the GBD to the closed-loop version of the value-optimal SND
problem.

1.1. Review of Cost-Optimal and Value-Optimal SND

Consider a process flow network with material balances $s_s = Ms_p$, where $s_p$ is a vector
of primary flow variables and $s_s$ are secondary variables (i.e., $s_s$ are completely determined
by $s_p$ and the material balances). Then, the vector of all material flows can be written as

$$s = \begin{bmatrix} s_p \\ s_s \end{bmatrix} = Cs_p \quad (1)$$

where $C = [I \ M^T]^T$ and the dimension of $s$ is $n \times 1$. Now let $\delta$ be the set of measurements
of all flow variables, each corrupted by independent measurement noise: $\delta = s + v$. Each
element of $v$ is zero mean and has a variance $\sigma^2_v$. Since each noise term is independent of
the others, the covariance matrix of $v$ is of a diagonal form.
Given the measurement vector, $\delta$, the optimal estimate of $s$ is

$$\hat{s} = \left[ C(C^T\Sigma^{-1}_v C)^{-1}C^T\Sigma^{-1}_v \right] \delta$$  \hspace{1cm} (2)

Notice that $\Sigma^{-1}_v = \text{diag}\{\sigma^{-2}_{v_1}, \sigma^{-2}_{v_2}, \ldots, \sigma^{-2}_{v_n}\}$. Thus, if $\sigma^{-2}_{v_i} = 0$ (or $\sigma^2_{v_i} \rightarrow \infty$), then the estimator will interpret as stream $i$ is unmeasured, even though $\delta_i$ remains in the formulation and the dimension of the matrices is unchanged. As such, the inverse variance will be used to indicate the presence or absence of a sensor at stream $i$: $\sigma^{-2}_{v_i} = \alpha_i \bar{\sigma}^{-2}_{v_i}$, where $\bar{\sigma}^2_{v_i}$ is the known variance of the sensor and $\alpha_i$ is the zero-one attendance variable.

The performance of the sensor network will be indicated by the estimation error: $q = s - \hat{s}$. While this error is unknowable, one can calculate the variance of each element of $q$:

$$\zeta_l = \rho_l \Sigma_q \rho_l^T, \hspace{0.5cm} l = 1 \ldots n_q$$  \hspace{1cm} (3)

where $\rho_l$ is the $l$th row of the $n_q$ identity matrix and

$$\Sigma_q = C(C^T\Sigma^{-1}_v C)^{-1}C^T$$  \hspace{1cm} (4)

Now define the performance criteria as $\zeta_l \leq \bar{\sigma}^2_{l}, l = 1 \ldots n_q$ where $\bar{\sigma}_l$ is the maximum allowable standard deviation of the estimation error. Then, one can employ the Schur Complement Theorem to arrive at the following set of convex Linear Matrix Inequality (LMI) constraint that is equivalent to (3)

$$\begin{bmatrix} \zeta_l & \rho_l C \\ C^T \rho_l^T & C^T \Sigma^{-1}_v C \end{bmatrix} \succeq 0, \hspace{0.5cm} \zeta_l \leq \bar{\sigma}^2_{l}, \hspace{0.5cm} l = 1 \ldots n_q$$  \hspace{1cm} (5)

where the $\succeq$ sign in (5) indicates that the matrix must be positive semi-definite. In addition,
one finds that

$$\Sigma_{\nu}^{-1} = \text{diag}\left\{ \alpha_1 \bar{\sigma}_{\nu_1}^{-2}, \alpha_2 \bar{\sigma}_{\nu_2}^{-2}, \ldots, \alpha_n \bar{\sigma}_{\nu_n}^{-2} \right\} = \sum_{i=1}^{n} \alpha_i \Theta_i$$  \tag{6}

where $\Theta_i = \bar{\sigma}_{\nu_i}^{-2} \rho_i^T \rho_i$. If the cost of a sensor at location $i$ is $c_i^{(s)}$, then the SDP version of the cost optimal SND problem for steady-state systems is stated as

$$\min_{\alpha_i \in \{0,1\}} \sum_{i=1}^{n} c_i^{(s)} \alpha_i \tag{7}$$

subject to

$$\zeta_l \leq \bar{\sigma}_l^2, \quad l = 1 \ldots n_q \tag{8}$$

$$\begin{bmatrix} \zeta_l & \rho_l C \\ C^T \rho_l^T & C^T \left( \sum_{i=1}^{n} \alpha_i \Theta_i \right) C \end{bmatrix} \succeq 0 \quad l = 1 \ldots n_q \tag{9}$$

The branch and bound method can be used to solve problem (7). However, it is noted that enforcement of the constraints of (9) requires the use of a SDP solver to solve the subproblem of each iteration of the branch and bound search. While SDP solvers will provide global solutions, they tend to converge slowly and require large amounts of memory. Thus, a reduction in the number of calls to the SDP solver is expected to provide a computational advantage.

In Bagajewicz et.al., [5], a method for assessing Downside Expected Financial Loss (DEFL) incurred by production loss was developed. Due to the inaccuracy of the estimate of a product stream flow rate, there is a finite probability that the estimate is above the target but in fact the real flow is below it. In such a situation it is assumed that the operator will not make any correction to production. As a result, production output will be less than the target and a financial loss will be incurred. The financial loss was determined to be

$$\text{DEFL} = 0.19947 K_s T \sigma_l$$

where $K_s$ is the cost of the product, $T$ is the time window of the analysis and $\sigma_l$ is the standard deviation of stream $l$. If one of the performance criteria is DEFL, then one must introduce a new set of variables, $\sigma_l$, and required each to be such that...
\[ \zeta_l \leq \sigma^2_l. \] As an extension of the cost optimal SND problem, the value-optimal SND problem is stated as

\[
\min_{\alpha_i \in \{0,1\}} \sum_{i=1}^{n} c_i^{(s)} \alpha_i + \sum_{l=1}^{n_q} 0.19947 K_s T \sigma_l \tag{10}
\]

s.t. \( \zeta_l \leq \sigma^2_l, \quad l = 1 \ldots n_q \) \tag{11}

\[
\begin{bmatrix}
\zeta_l \\
C^T \rho_l \\
C^T \left( \sum_{i=1}^{n} \alpha_i \Theta_i \right) C
\end{bmatrix} \succeq 0 \quad l = 1 \ldots n_q \tag{12}
\]

1.2. Review of the Generalized Benders Decomposition

Consider the optimization problem

\[
\min_{x \in X, y \in Y} F(x, y) \text{ s.t. } G(x, y) \leq 0 \tag{13}
\]

where \( y \) is a vector of complicating variables in that for a given \( \bar{y} \in V \triangleq \{ y \mid G(x, y) \leq 0 \text{ for some } x \in X \} \), the following primal problem is easily solved

\[
\min_{x \in X} F(x, \bar{y}) \text{ s.t. } G(x, \bar{y}) \leq 0 \tag{P}
\]

If \( X \) is a convex set and both \( F(x, y) \) and \( G(x, y) \) are convex in \( x \), then the following relaxed master problem will converge to the global solution of (13), see Geoffrion [12] for details.

\[
\min_{y \in Y, y_0} y_0 \tag{14}
\]

\[
L^* (y, \mu^{(k_1)}) \leq y_0 \quad k_1 = 1 \ldots K_1 \tag{15}
\]

\[
L_* (y, \lambda^{(k_2)}) \leq 0 \quad k_2 = 1 \ldots K_2 \tag{16}
\]
where $L^*$ and $L_*$ are defined as

\begin{align}
L^*(y, \mu) &= \min_{x \in X} \left\{ F(x, y) + \mu^T G(x, y) \right\} \\
L_*(y, \lambda) &= \min_{x \in X} \left\{ \lambda^T G(x, y) \right\}
\end{align}

and $\mu^{(k_1)}$ and $\lambda^{(k_2)}$ are selected by the following procedure:

1. Select a point $\bar{y} \in V \cap Y$. Solve the primal problem (P) and obtain its solution, $x^*$, along with the multipliers associated with the constraints $G(x, \bar{y}) \leq 0$, $\mu^*$. Set $K_1 = 1$, $K_2 = 0$, $\mu^{(1)} = \mu^*$, $UBD = F(x^*, \bar{y})$ and determine $L^*(y, \mu^{(1)})$.

2. Solve the relaxed master problem (14) and obtain its global solution, $(y^*_0, y^*)$. Set $LBD = y^*_0$. If $UBD \leq LBD + \epsilon$, the problem has converged.

3. Set $\bar{y} = y^*$. If the primal problem (P) is feasible go to step 3a; if infeasible go to 3b.

3a. Solve the primal problem and obtain the solution and multipliers $(x^*, \mu^*)$. Set $K_1 = K_1 + 1$, $\mu^{(K_1)} = \mu^*$ and determine $L^*(y, \mu^{(K_1)})$. If $F(x^*, \bar{y}) < UBD$ set $UBD = F(x^*, \bar{y})$. Return to step 2.

3b. Solve the following problem (denoted Problem Q):

\begin{equation}
\max_{\lambda \in \Lambda} \left\{ \min_{x \in X} \left\{ \lambda^T G(x, \bar{y}) \right\} \right\}
\end{equation}

where $\Lambda = \{ \lambda \geq 0 \mid \lambda^T \mathbf{1} = 1 \}$ and $\mathbf{1} = [1 \ldots 1]^T$. Obtain its solution $(x^*, \lambda^*)$. Set $K_2 = K_2 + 1$, $\lambda^{(K_2)} = \lambda^*$, determine $L_* (y, \lambda^{(K_2)})$ and return to step 2.

As shown in references [10], [13] and [14] that the solution to Problem Q can be found from the following problem — as its solution along with its associated multipliers $(x^*, \lambda^*)$. Specifically, $\lambda^*$ are the optimal dual variables associated with the constraints $G(x, \bar{y}) - \pi \mathbf{1} \leq 0$.

\begin{equation}
\min_{x \in \mathcal{X}, \pi} \left\{ \pi \right\} \text{ s.t. } G(x, \bar{y}) - \pi \mathbf{1} \leq 0
\end{equation}
If the solution to (Q) is such that \( \pi^* \leq 0 \), then the primal problem would have been feasible. If \( F \) and \( G \) are linearly separable (i.e., \( F(x, y) = F_1(x) + F_2(y) \) and \( G(x, y) = G_1(x) + G_2(y) \)), then both \( L^* \) and \( L_* \) can be calculated as explicit functions of \( y \), [12].

2. Application of GBD to Value-optimal SND for Steady-state Systems

To employ the GBD approach, the attendance variable \( \alpha_i \) must be split into two parts - one part for the cost of the component, \( \alpha_i^{(cc)} \), and a second for its impact on system performance, \( \alpha_i^{(sp)} \), [11]. Using an extension of Theorem 1 of [11], the equivalent form of problem (10) is obtained as

\[
\min_{\alpha_i^{(cc)}, \alpha_i^{(sp)}} \sum_{i=1}^{n} c_i^{(s)} \alpha_i^{(cc)} + \sum_{i=1}^{n_q} 0.19947K_s T \sigma_l \tag{20}
\]

s.t. \( \alpha_i^{(cc)} \in \{0, 1\} \)

\( \alpha_i^{(sp)} - \alpha_i^{(cc)} \leq 0, \quad i = 1 \ldots n \) \tag{21}

\( \zeta_l \leq \sigma_l^2, \quad l = 1 \ldots n_q \) \tag{22}

\[
\begin{bmatrix}
\zeta_l \\
C^T \rho_l \rho_l^T \sum_{i=1}^{n} \alpha_i^{(sp)} \Theta_i C
\end{bmatrix} \succeq 0, \quad l = 1 \ldots n_q \tag{23}
\]

\[
0 \leq \alpha_i^{(sp)} \leq 1, \quad i = 1 \ldots n \tag{24}
\]

It is clearly observed that constraints (24) and (25) are convex. Thus, the non-complicating variables, \( x \), are selected as \( \alpha_i^{(sp)} \) and \( \zeta_l \), and the set \( X \) is defined as \( \{\alpha_i^{(sp)}, \zeta_l | (24) \text{ and } (25) \text{ are satisfied}\} \). Then, the complicating variables, \( y \), should be selected as \( \alpha_i^{(cc)} \) and \( \sigma_l \), and the set \( Y \) is defined as \( \{\alpha_i^{(cc)}, \sigma_l | (21) \text{ is satisfied}\} \). The connecting constraints \( G(x, y) \leq 0 \) consist of constraints (22) and (23). Clearly, \( F(x, y) \) and \( G(x, y) \) and \( X \) are convex with respect to \( x \) and thus a globally optimal solution will be guaranteed. In addition, \( F(x, y) \) and \( G(x, y) \)
are linearly separable, indicating that $L^*$ and $L_*$ can be determined as explicit functions of $y$. Given these selections, it is noted that the objective is a function of only the complicating variables $y$ (i.e. $F(x, y) = F_2(y)$). Hence, the primal problem is just a feasibility problem:

$$\min_{\alpha_i^{(sp)}, \bar{\alpha}_i} \sum_{i=1}^{n} c_i^{(s)} \alpha_i^{(cc)} + \sum_{i=1}^{n_q} 0.19947K_sT \bar{\sigma}_l$$

s.t. $\alpha_i^{(sp)} - \bar{\alpha}_i^{(cc)} \leq 0, \ i = 1 \ldots n$

$$\zeta_l - \bar{\sigma}_l^2 \leq 0, \ l = 1 \ldots n_q$$

(24), (25)

The first step of the GBD algorithm requires $\bar{y} \in V \cap Y$. An obvious choice is to select $\alpha_i^{(cc)} = 1$ for all $i$, and $\sigma_l$ to be a sufficiently large number for all $l$. If the resulting primal problem is infeasible, then terminate, as the original optimization problem is infeasible. If the primal is feasible then the optimal multipliers, $\mu^*$, will be zero. In this case, $\text{UBD}=F_2(\bar{y})$ and $L^*(y, \mu^{(1)}) = F_2(y)$. Then, in the first execution of step 2, problem (14) with $(K_1 = 1$ and $K_2 = 0)$, gives the solution $y^* = 0$ and $\text{LBD}=0$. In the first execution of step 3 with $\bar{y} = 0$, the primal is obviously infeasible. Thus, the following version of problem $Q$ should be solved to obtain the solution $x^*$ and the associated multiplier $\lambda^*$.

$$\min_{\alpha_i^{(sp)}, \bar{\alpha}_i, \pi} \{\pi\} \text{ s.t.}$$

$$\alpha_i^{(sp)} - \bar{\alpha}_i^{(cc)} - \pi \leq 0, \ i = 1 \ldots n$$

$$\zeta_l - \bar{\sigma}_l^2 - \pi \leq 0, \ l = 1 \ldots n_q$$

(24), (25)

Then, returning to step 2 to solve (14), with $K_2$ incremented by 1, one will need to calculate $L_s(y, \lambda^{(1)})$ where $\lambda^{(1)}$ are the optimal multipliers of (Q1) associated with (26) and (27). If $\lambda$ is defined as $\lambda = [\gamma_1 \ \gamma_2 \ \ldots \ \gamma_n \ \omega_1 \ \omega_2 \ \ldots \ \omega_{n_q}]^T$, then the solution and optimal multipliers
\((x(k_2), \lambda(k_2))\) are denoted as \((\alpha_i^{(sp,k_2)}, \zeta_i^{(k_2)}, \gamma_i^{(k_2)}, \omega_i^{(k_2)})\) and \(L_s(y, \lambda(k_2)) = \beta_{k_2} - \sum_i \gamma_i^{(k_2)} \alpha_i^{(cc)} - \sum_l \omega_i^{(k_2)} \sigma_l^2\). Therefore, the relaxed master problem is found to be:

\[
\min_{\alpha_i^{(cc)} \in \{0,1\}, \delta_i} \left\{ \sum_{i=1}^{n} \alpha_i^{(s)} \alpha_i^{(cc)} + \sum_{l=1}^{n} 0.19947K_sT \sigma_l \right\} \quad \text{s.t.} \quad \text{(RM1)}
\]

\[
\beta_{k_2} - \sum_{i=1}^{n} \gamma_i^{(k_2)} \alpha_i^{(cc)} - \sum_{l=1}^{n} \omega_i^{(k_2)} \sigma_l^2 \leq 0, \ k_2 = 1 \ldots K_2
\]

If at any iteration the solution to (RM1) is such that the primal problem, (P1), is feasible, then the solution to the primal will have an objective value equal to the previous relaxed master. In this case, the algorithm terminates since UBD=LBD.

To summarize, the GBD algorithm implements an iteration between two subproblems: the relaxed master problem (RM1) and problem Q (Q1). Problem Q is a convex semi-definite program without integer variables, and requires the use of a SDP solver. The SDP solver used must be capable of generating the optimal dual variables \((\gamma_i^{(k_2)}, \omega_i^{(k_2)})\) associated with constraints \(G(x, y) - \pi_1 \leq 0\).

Since constraint (28) is nonconvex (due to the term \(\sigma_l^2\)), the relaxed master is a Mixed Integer Nonlinear Program (MINLP). As such, the branch and bound method must be used to solve (RM1), which will greatly increase computational effort. The following subsection will illustrate how to remove this non-convexity from the relaxed master to arrive at a MILP.

### 2.1. Convexification of the relaxed master problem

Consider the following restatement of the value-optimal optimal SND problem

\[
\min_{x \in X, y \in Y} F(x, y) \quad \text{s.t.} \quad G'(x, y) \leq 0
\]  

where \(G'(x, y)\) consists of \(\alpha_i^{(sp)} - \alpha_i^{(cc)}\) and \(\sqrt{\xi_l} - \sigma_l\).

**Theorem 1** Problem (29) is equivalent to problem (20), and no duality gap exists when
implementing GBD.

Proof. For all $\zeta \geq 0$ and $\sigma \geq 0$, $\sqrt{\zeta} \leq \sigma$ if and only if $\zeta \leq \sigma^2$. Moreover, for all $\bar{\sigma} \geq 0$ and $\bar{\alpha}^{(cc)}$, $X_{G'} = \{ \zeta \geq 0, 0 \leq \alpha^{(sp)} \leq 1 \mid G'(x, \bar{y}) \leq 0 \} = \{ \zeta \geq 0, 0 \leq \alpha^{(sp)} \leq 1 \mid G(x, \bar{y}) \leq 0 \}$ is a convex region. $F(x, y)$ is also convex in $x$. Thus no duality gap exits when implementing the GBD approach. \hfill \blacksquare

In this case, problem (Q) will need to be solved with $G(x, \bar{y}) - \pi_1 \leq 0$ replaced by $G'(x, \bar{y}) - \pi_1 \leq 0$, or $\bar{\alpha}_i^{(sp)} - \bar{\alpha}_i^{(cc)} - \pi_1 \leq 0$ and $\sqrt{\zeta_i} - \bar{\sigma}_i - \pi_1 \leq 0$. Clearly, this second group of constraints is non-convex. To address this issue, consider problem (19) using $G'(x, \bar{y})$:

$$\max_{\gamma_i \geq 0, \omega_l \geq 0} \left\{ \min_{x \in X} \left\{ \sum_i \gamma_i (\alpha_i^{(sp)} - \bar{\alpha}_i^{(cc)}) + \sum_l \omega_l (\sqrt{\zeta_l} - \bar{\sigma}_l) \right\} \right\} \tag{Q2}$$

s.t. $\sum_i \gamma_i + \sum_l \omega_l = 1$

The following theorem provides a method of determining the solution to this non-convex problem via the solution to a convex problem.

Theorem 2 The solution to problem (Q2), $(\alpha_i^{(sp)}^*, \gamma_i^*, \zeta_l^*, \omega_l^*)$, is given by $\alpha_i^{(sp)} = \alpha_i^{(sp)*}$, $\gamma_i^* = \gamma_i^*/a$, $\zeta_l^* = \zeta_l^*$ and $\omega_l^* = \omega_l^*/\sqrt{\zeta_l^*}a$ where $a = \sum_i \gamma_i^* + \sum_l \omega_l^*/\sqrt{\zeta_l^*}$ and $(\alpha_i^{(sp)*}, \gamma_i^*, \zeta_l^*, \omega_l^*)$ is the solution to:

$$\max_{\gamma_i' \geq 0, \omega_l' \geq 0} \left\{ \min_{x' \in X} \left\{ \sum_i \gamma_i' (\alpha_i^{(sp)} - \bar{\alpha}_i^{(cc)}) + \sum_l \omega_l' (\zeta_l' - \sqrt{\zeta_l' \bar{\sigma}_l}) \right\} \right\} \tag{Q3}$$

s.t. $\sum_i \gamma_i' + \sum_l \omega_l' = 1$

Proof. For any positive constant $a$, the solution to problem (Q2) is $\lambda^* = \hat{\lambda}^*/a$ where $\hat{\lambda}^*$ is
the solution to
\[
\begin{align*}
\max_{\lambda \geq 0} \left\{ \min_{x \in X} \left\{ \sum_i \tilde{\gamma}_i (\alpha_i^{(sp)} - \tilde{\alpha}_i^{(cc)}) + \sum_l \tilde{\omega}_l (\sqrt{\zeta_l} - \tilde{\sigma}_l) \right\} \right\} \\
\text{s.t.} \quad \sum_i \tilde{\gamma}_i + \sum_l \tilde{\omega}_l = a
\end{align*}
\]
\[
\begin{align*}
= \max_{\lambda \geq 0} \left\{ \min_{x \in X} \left\{ \sum_i \tilde{\gamma}_i (\alpha_i^{(sp)} - \tilde{\alpha}_i^{(cc)}) + \sum_l \frac{\tilde{\omega}_l}{\sqrt{\tilde{\zeta}_l}} (\sqrt{\zeta_l} - \sqrt{\tilde{\sigma}_l}) \right\} \right\} \\
\text{s.t.} \quad \sum_i \tilde{\gamma}_i + \sum_l \frac{\tilde{\omega}_l}{\sqrt{\tilde{\zeta}_l}} = a
\end{align*}
\]
\[
\begin{align*}
= \max_{\lambda' \geq 0} \left\{ \min_{x' \in X} \left\{ \sum_i \tilde{\gamma}'_i (\alpha_i^{(sp)} - \tilde{\alpha}_i^{(cc)}) + \sum_l \omega'_l (\sqrt{\zeta'_l} - \sqrt{\tilde{\sigma}_l}) \right\} \right\} \\
\text{s.t.} \quad \sum_i \tilde{\gamma}'_i + \sum_l \omega'_l \sqrt{\zeta'_l} = a
\end{align*}
\]

Since \( a \) can be selected as any positive constant, the constraint \( \sum_i \tilde{\gamma}'_i + \sum_l \omega'_l \sqrt{\zeta'_l} = a \) may be removed as long as \( \sum_i \tilde{\gamma}'_i + \sum_l \omega'_l \sqrt{\zeta'_l} \) is guaranteed to be positive. Since \( \zeta'_l > 0 \) for all \( l \), this guarantee is achieved by enforcing \( \sum_i \tilde{\gamma}'_i + \sum_l \omega'_l = 1 \). Thus, given the solution, \((\alpha_i^{(sp)*}, \zeta'^*, \lambda'^*)\) of (Q3), the solution to (Q2) is \( \alpha_i^{(sp)*} = \alpha_i^{(sp)*}, \gamma'_l = \gamma'_l / a = \gamma'^* / a, \zeta'_l = \zeta'^* \) and \( \omega'_l = \tilde{\omega}_l / a = \omega'^* \sqrt{\zeta'^* / a} \). ■

Given Theorem 2, if the solution to problem (Q3) is \((\alpha_i^{(sp,k_2)}, \zeta'^{(k_2)}, \gamma'^{(k_2)}, \omega'^{(k_2)})\), then the appropriate relaxed master problem is easily obtained as
\[
\begin{align*}
\min_{\alpha^{(cc)} \in \{0,1\}, \sigma} \left\{ \sum_i n \epsilon_i^{(s)} \alpha_i^{(cc)} + \sum_l n \epsilon_l \right\} \\
\text{s.t.} \quad \beta_k - \sum_i \gamma_i^{(k_2)} \alpha_i^{(cc)} - \sum_l \omega_i^{(k_2)} \sigma_l \leq 0, \quad k = 1 \ldots K_2
\end{align*}
\]
where \( \beta_k = (\sum_i \gamma_i^{(k_2)} \alpha_i^{(sp,k_2)} + \sum_l \omega_i^{(k_2)} \sqrt{\zeta_l^{(k_2)})}) / a, \quad \gamma_i^{(k_2)} = \gamma_i^{(k_2)} / a, \quad \omega_i^{(k_2)} = \omega_i^{(k_2)} / \sqrt{\zeta_l^{(k_2)}} / a, \quad a = \sum_i \gamma_i^{(k_2)} + \sum_l \omega_l^{(k_2)} \sqrt{\zeta_l^{(k_2)}}. \) Furthermore, the solution to problem (Q3) can be obtained
from

\[
\begin{align*}
\min & \quad \{\pi\} \\
\text{s.t.} & \quad \alpha_i^{(sp)} - \bar{\alpha}_i^{(cc)} - \pi \leq 0, \quad i = 1 \ldots n \\
& \quad \zeta'_l - \sqrt{\zeta'_l \bar{\sigma}_l} - \pi \leq 0, \quad l = 1 \ldots n_q \\
\end{align*}
\]

(24), (25)

\[
\begin{align*}
= \min & \quad \{\pi\} \\
\text{s.t.} & \quad \alpha_i^{(sp)} - \bar{\alpha}_i^{(cc)} - \pi \leq 0, \quad i = 1 \ldots n \\
& \quad \zeta'_l - \tau_l \bar{\sigma}_l - \pi \leq 0, \\
& \quad \begin{bmatrix} \zeta'_l & \tau_l \\ \tau_l & 1 \end{bmatrix} \succeq 0, \quad l = 1 \ldots n_q \\
\end{align*}
\]

(24), (25)

Clearly this problem is convex and a global solution can be readily obtained.

Summarizing the GBD algorithm, it implements an iteration between the convexified relaxed master problem and problem (Q5) (see Figure 1). The convexified relaxed master

\begin{center}
\textbf{Covexified Relaxed Master (RM2)}
\end{center}

\begin{center}
\text{(MILP solved with GUROBI)}
\end{center}

\begin{center}
\begin{array}{c}
\bar{\alpha}_i^{(cc)} \\
\bar{\sigma}_l \\
\end{array}
\end{center}

\begin{center}
\begin{array}{c}
\lambda_l^{(k_1)}, \omega_l^{(k_2)} \\
\beta_{k_2} \\
\end{array}
\end{center}

\begin{center}
\textbf{Problem (Q5)}
\end{center}

\begin{center}
\text{(SDP solved with MOSEK)}
\end{center}

\begin{center}
Figure 1: Illustration of GBD algorithm.
\end{center}

(RM2) is a mixed integer linear program (MILP) without any matrix inequalities. Thus it can be solved using a variety of MILP solvers. It is also noted that the convexified relaxed

13
master at each iteration is the same as the previous except that a new linear constraint is added. In the examples of this paper, the MILP solver GUROBI, [17], was used. Problem (Q5) is a convex semi-definite program without integer variables, and can be solved by a SDP solver. As mentioned before, the SDP solver used must be capable of generating the optimal dual variables \( (\gamma_i^{(k_2)}, \omega_i^{(k_2)}) \) associated with constraints \( G(x,y) - \pi_1 \leq 0 \). The SDP solver MOSEK, [15], was employed. Both solvers were used via YALMIP, [16], in a MATLAB environment.

2.2. Case Studies for Steady-State Systems

This subsection will demonstrate the computational advantage of the GBD approach compared to the brand and bound method. All examples are solved on 64-bit windows PC with Intel 2.5 GHz CPU and 8 GB RAM.

Example 1: Consider the flow diagram of Figure 2, [1]. If \( s_2 \) and \( s_3 \) are selected as primal variables, the matrix \( C \) of equation (1) is \( C = [C_0^T C_0^T C_0^T]^T \) where

\[
C_0 = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}^T
\]  

(30)

The nominal flow rates are assumed to be \( s = [s_0^T s_0^T s_0^T] \) where \( s_0 = [150.1, 52.3, 97.8, 97.8]^T \) (kg/min). The sensor options are 1%, 2%, or 3% relative errors. The resulting measurement precisions and sensor costs are presented in Table 1.

The objective is to minimize the sum of the sensor cost and the DEFL associated with
Table 1: Precision, relative error and sensor cost for Example 1.

<table>
<thead>
<tr>
<th>Sensor #</th>
<th>Stream</th>
<th>Relative Error (%)</th>
<th>Precision ($\sigma_{vi}$)</th>
<th>Flowmeter Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.501</td>
<td>2500</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0.523</td>
<td>2500</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0.978</td>
<td>2500</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0.978</td>
<td>2500</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3.002</td>
<td>1500</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2</td>
<td>1.046</td>
<td>1500</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>2</td>
<td>1.956</td>
<td>1500</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1.956</td>
<td>1500</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>3</td>
<td>4.503</td>
<td>800</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>3</td>
<td>1.569</td>
<td>800</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>3</td>
<td>2.934</td>
<td>800</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>3</td>
<td>2.934</td>
<td>800</td>
</tr>
</tbody>
</table>

steams 1 and 4. The parameter $K_s$ is assumed to be $3600/year, and $T$ is assumed to be 5 years.

Both the branch and bound and the GBD methods generate the same solution: 1% sensor placed at streams 2 and 3, the standard deviation of the estimation error on streams 1 and 4 are found to be 1.1163 and 1.0205, with an optimal objective function value of $12,672$. The GBD approach achieves a 98.6% reduction in computational effort, as indicated in Table 2.

Table 2: Comparison of computational efforts for Example 1.

<table>
<thead>
<tr>
<th>Method</th>
<th>Iterations</th>
<th>SDP Calls</th>
<th>Time (sec)</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Branch and Bound</td>
<td>612</td>
<td>1224</td>
<td>492</td>
<td>-</td>
</tr>
<tr>
<td>GBD</td>
<td>11</td>
<td>11</td>
<td>7</td>
<td>98.6%</td>
</tr>
</tbody>
</table>

Figure 3: Process flow diagram for Example 2, [19]

Example 2: This example is adapted from Example 3 of [18], which was adapted from
Consider the 24 stream network of Figure 3. The flow rate and the cost of sensors for each stream are listed in Table 3. The precision of sensors are assumed to be 2.5%. The design objective is to minimize the sum of sensor cost and the DEFL associated with steams 3, 10, 16, 17, 20 and 24. The parameter $K_s$ is assumed to be 10, and $T$ is assumed to be 5.

Table 3: Flow rate and flowmeter cost of each stream for Example 2.

<table>
<thead>
<tr>
<th>Stream</th>
<th>Flow Rate</th>
<th>Flowmeter Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>140</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>130</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>45</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>13</td>
</tr>
<tr>
<td>11</td>
<td>80</td>
<td>17</td>
</tr>
<tr>
<td>12</td>
<td>40</td>
<td>13</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>15</td>
<td>90</td>
<td>17</td>
</tr>
<tr>
<td>16</td>
<td>100</td>
<td>19</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>17</td>
</tr>
<tr>
<td>18</td>
<td>135</td>
<td>18</td>
</tr>
<tr>
<td>19</td>
<td>45</td>
<td>17</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>15</td>
</tr>
<tr>
<td>21</td>
<td>80</td>
<td>15</td>
</tr>
<tr>
<td>22</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>23</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>24</td>
<td>45</td>
<td>13</td>
</tr>
</tbody>
</table>

The branch-and-bound and GBD approach both obtain the same solution, which requires sensors at streams 4, 8, 10, 16, 20, 23 and 24, with the resulting standard deviation of estimation error on streams 3, 10, 16, 17, 20 and 24 being 2.7028, 2.5000, 2.4971, 0.1292, 0.7496 and 1.1242, respectively. The optimal objective function value is $188. Comparison of the computational efforts for the two approaches is given in Table 4 and illustrates a computational effort reduction of almost 80%.
Table 4: Comparison of computational efforts for Example 2.

<table>
<thead>
<tr>
<th>Method</th>
<th>Iterations</th>
<th>SDP Calls</th>
<th>Time (sec)</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Branch and Bound</td>
<td>2222</td>
<td>4444</td>
<td>2960</td>
<td>-</td>
</tr>
<tr>
<td>GBD</td>
<td>257</td>
<td>257</td>
<td>607</td>
<td>79.5%</td>
</tr>
</tbody>
</table>

3. Value-Optimal SND for Dynamic Process

This section illustrates application of the GBD to the value-optimal SND problem for closed-loop processes. The first subsection presents a short review of the nonconvex mixed integer programming formulation for the value-optimal SND problem, while the second describes the application of GBD to this class of problems and demonstrates the computational advantage of GBD compared to the branch and bound approach.

3.1. Review of Value-Optimal SND for Dynamic Process

Consider a dynamic process model: \( \dot{s} = f(s, m, p) \), \( q = h(s, m, p) \), and \( q^{\text{min}} \leq q \leq q^{\text{max}} \), where \( s, m \) and \( q \) are the state, manipulated and performance variables, each being a column vector with dimension of \( n_s, n_m \) and \( n_q \), respectively, and \( p \) is a white noise process with a mean \( \bar{p} \) and spectral density of \( \Sigma_p \). The Optimal Steady-State Operating (OSSOP) can be obtained from the following optimization problem:

\[
\min_{s^{ss}, m^{ss}} \{ g(q^{ss}) \} \quad \text{s.t.} \quad \begin{align*}
0 &= f(s^{ss}, m^{ss}, \bar{p}) \\
q^{ss} &= h(s^{ss}, m^{ss}, \bar{p}) \\
q^{\text{min}} &\leq q^{ss} \leq q^{\text{max}}
\end{align*}
\]  

(31)

where \( g(\cdot) \) is the operating cost function. The solution to problem (31) is denoted as \( s^{\text{OSSOP}}, m^{\text{OSSOP}}, q^{\text{OSSOP}} \). If the process is operated at the OSSOP, then the actual non-steady-state disturbances (i.e., \( p \neq \bar{p} \)) will make \( q \) deviate from \( q^{\text{OSSOP}} \) and cause significant constraint violations. Thus, a Backed-off Operating Point (BOP) should be selected that is some distance from the operating constraints, but still close to the OSSOP, as indicated in Figure...
4. The BOP is denoted as \((s^{BOP}, m^{BOP}, p^{BOP})\), and must satisfy the steady-state process model:

\[0 = f(s^{BOP}, m^{BOP}, \bar{p}), q^{BOP} = h(s^{BOP}, m^{BOP}, \bar{p})\]  
(32)

\[q^{min} \leq q^{BOP} \leq q^{max}\]  
(33)

Figure 4: Illustration of the MBOP problem.

To characterize the Expected Dynamic Operation Region (EDOR), we start with a linearization of the nonlinear dynamic model \(\dot{s} = A\tilde{s} + B\tilde{m} + G\tilde{p}, \tilde{q} = D_s\tilde{s} + D_m\tilde{m}\) where the deviation variables \((\tilde{s}, \tilde{m}, \tilde{p}, \tilde{q})\) are defined as \(\tilde{s} = s - s^{BOP}, \tilde{m} = m - m^{BOP}, \tilde{p} = p - p^{BOP}\), and \(\tilde{q} = q - q^{BOP}\). The manipulated variable is a linear feedback of the state estimate: \(\tilde{m} = L\hat{s}\), where \(L\) is to be determined and \(\hat{s}\) is the optimal state estimate of \(\tilde{s}\). If the estimation error is \(e = \tilde{s} - \hat{s}\), then its covariance \(\Sigma_e\) is calculated as:

\[A\Sigma_e + \Sigma_e A^T + G\Sigma_p G^T - \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e = 0\]  
(34)

where \(C\) and \(\Sigma_v\) are from the measurement equation \(\tilde{s} = C\hat{s} + v\) where \(v\) is a zero-mean
white noise with spectral density of $\Sigma_v$. Note that $\Sigma_v^{-1} = \sum_{i=1}^{n} \alpha_i \Theta_i$ where $\Theta_i$ is as defined in (6). Then, using the covariance analysis of [4], the variance of $l^{th}$ output of $q$ is denoted as $\zeta_l$ and is equal to the $l^{th}$ diagonal of $\Sigma_q$:

$$
\zeta_l = \rho_l \Sigma_q \rho_l^T = \rho_l (D_s \Sigma_e D_s^T + (D_s + D_m L)(\Sigma_s - \Sigma_e)(D_s + D_m L)^T) \rho_l^T, \ l = 1 \ldots n_q
$$

(35)

where $\rho_l$ is defined as the $l^{th}$ row of an $n_q \times n_q$ identity matrix, and $\Sigma_s$ is the covariance matrix associated with $s$ and is found as the positive define solution to

$$
A \Sigma_s + \Sigma_s A^T + G \Sigma_w G^T + BL(\Sigma_s - \Sigma_e) + (\Sigma_s - \Sigma_e)L^T B^T = 0
$$

(36)

One can impose upper bounds to the EDOR in each direction by enforcing the constraints $\sqrt{\zeta_l} \leq \sigma_l$. However, to guarantee the resulting BOP and its associated EDOR is contained in the region defined by $q_{\min}$ and $q_{\max}$, the following constraints must be enforced: $\sqrt{\zeta_l} \leq q_{l, BOP} - q_{l, \min}$ and $\sqrt{\zeta_l} \leq q_{l, \max} - q_{l, BOP}$, $l = 1 \ldots n_q$. These can be written equivalently as

$$
\sigma_l \leq q_{l, BOP} - q_{l, \min}, \quad \sigma_l \leq q_{l, \max} - q_{l, BOP}, \ l = 1 \ldots n_q
$$

(37)

$$
\sqrt{\zeta_l} \leq \sigma_l, \ l = 1 \ldots n_q
$$

(38)

The value-optimal SND for closed-loop dynamic processes is then a result of combining the steady-state perspective (BOP) and the dynamic perspective (EDOR).

$$
\min_{\sigma_{BOP} > 0, \zeta_{BOP} > 0, \alpha_i \in \{0,1\}} \left\{ g(q_{BOP}) + \sum_{i=1}^{n} c_i^{(s)} \alpha_i \right\}
$$

s.t. (32), (33), (34), (35), (36), (37), (38)

Theorem 1 of Peng and Chmielewski [7] indicates that the nonlinear equalities of (34)-(36)
hold if and only if the LMIs of (40)-(42) are satisfied. As a result the SND can be stated as:

\[
\min_{\sigma BOP, m BOP, q BOP, \alpha_i \in \{0,1\}} \left\{ g(q^{BOP}) + \sum_{i=1}^{n} c_i^{(s)} \alpha_i \right\} \text{ s.t. (32), (33), (37), (38)}
\]

\[
\begin{bmatrix}
C^T \left( \sum_{i=1}^{n} \alpha_i \Theta_i \right) C - A^T M_2 - M_2 A & M_2 G \\
G^T M_2 & \Sigma_w^{-1}
\end{bmatrix} \succeq 0
\]

\[
\begin{bmatrix}
\zeta_l & \rho_l (D_s M_0 + D_m M_1) & \rho_l D_s \\
(D_s M_0 + D_m M_1)^T \rho_l & M_0 & I \\
D_s^T \rho_l & I & M_2
\end{bmatrix} \succeq 0, \ l = 1 \ldots n_q
\]

\[
(AM_0 + BM_1) + (AM_0 + BM_1)^T G \Sigma_w G^T \preceq 0
\]

\[
\zeta_l \leq (q_l^{\max} - q_l^{\min})^2 / 4, \ l = 1 \ldots n_q
\]

Note the addition of (43). While (43) is redundant with (37), it will be useful in the following section. If \(\alpha_i^*, M_0^*, M_1^*, M_2^*\) is part of the optimal solution, then a feasible controller under this configuration can be calculated as \(L = M_1^* (M_0^* - (M_2^*)^{-1})^{-1}\). It should be noted that the formulation of the value-optimal SND problem for a discrete-time dynamic process can be obtained through a simple modification of constraints (32), (33), (40) and (42), [4].

### 3.2. Application of GBD

Similar to the steady-state SND case, the attendance variable, \(\alpha_i\) is split into two parts: the component associated with cost, \(\alpha_i^{(cc)}\), and the component impacting system performance,
\( \alpha_i^{(sp)} \). Then the problem (39) can be equivalently expressed as:

\[
\min_{\mathbf{s}^{BOP}, \mathbf{m}^{BOP}, q^{BOP}, \alpha_i^{(cc)}, \alpha_i^{(sp)}} \left\{ g(q^{BOP}) + \sum_{i=1}^{n} c_i^{(s)} \alpha_i^{(cc)} \right\} \tag{44}
\]

s.t. (32), (33), (37)

\( \alpha_i^{(cc)} \in \{0, 1\} \tag{45} \)

\( \alpha_i^{(sp)} - \alpha_i^{(cc)} \leq 0, \ i = 1 \ldots n \tag{46} \)

\( \sqrt{\sigma_l} \leq \sigma_l, \ l = 1 \ldots n_q \tag{47} \)

\[
\begin{bmatrix}
C^T \left( \sum_{i=1}^{n} \alpha_i^{(sp)} \Theta_i \right) C - A^T M_2 - M_2 A \\
G^T M_2 \\
\Sigma_w^{-1}
\end{bmatrix} \preceq 0 \tag{48}
\]

(41), (42), (43)

\( 0 \leq \alpha_i^{(sp)} \leq 1, \ i = 1 \ldots n \tag{49} \)

Based on this reformulation the complicating variables, \( y \), should be selected as \( \alpha_i^{(cc)} \), \( s^{BOP} \), \( m^{BOP} \), \( q^{BOP} \) and \( \sigma_l \), and the set \( Y \) is the constraints (32), (37) and (45). The non-complicating variables, \( x \), are selected as \( \alpha_i^{(sp)} \), \( \zeta \), \( M_0 \), \( M_1 \) and \( M_2 \), and the set \( X \) is constraints (41), (42), (43), (48) and (49). Finally, the connecting constraints, \( G'(x, y) \leq 0 \), are (46) and (47). Again, \( G'(x, y) \) is linearly separable, and thus \( L^* \) and \( L_* \) can be determined as explicit functions of \( y \). Using Theorem 1 again, it is found that the GBD algorithm is guaranteed to find a global solution. However, similar to the results of Section 2.1, use of \( G'(x, y) \) will result in Problem Q being non-convex. Fortunately, Theorem 2 can be re-applied such that the solution to problem Q is \( \gamma_i^{(k_2)} = \gamma_i^{(k_2)} / a \), \( \omega_i^{(k_2)} = \omega_i^{(k_2)} \sqrt{\zeta_l^{(k_2)}} / a \), and \( a = \sum_i \gamma_i^{(k_2)} + \sum_l \omega_i^{(k_2)} \sqrt{\zeta_l^{(k_2)}} \) where \( \zeta_l^{(k_2)} \), \( \gamma_i^{(k_2)} \) and \( \omega_i^{(k_2)} \) are the solution and multipliers
from

\[
\min_{\pi, \alpha^{(sp)}_i, \pi_l > 0} \pi \quad \text{(Q6)}
\]

\[
M_0 \geq 0, M_2 \geq 0, M_i \geq 0
\]

s.t. \quad \alpha^{(sp)}_i - \alpha^{(cc)}_i - \pi \leq 0, \quad i = 1 \ldots n \quad \text{(50)}

\[
\zeta'_l - \tau_l \bar{\sigma}_l - \pi \leq 0, \quad \begin{bmatrix} \zeta'_l & \tau_l \end{bmatrix} \succeq 0, \quad l = 1 \ldots n_q \quad \text{(51)}
\]

(41), (42), (43), (48), (49)

Figure 5: Furnace reactor system.
3.3. Case Studies for Closed-loop Systems

Example 3: Consider the furnace reactor system of Figure 5. The system matrices are

\[
A = \begin{bmatrix}
-10 & 0 & 0 & 0 & 0 \\
8000 & -8000 & 0 & 0 & 0 \\
0 & 2000 & -1500 & 0 & 0 \\
0 & 0 & 0 & -5000 & 0 \\
0 & 0 & 0 & 0 & -5000
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
-75 & 75000 & 0 \\
-25 & 0 & 0 \\
0 & -8500 & 8.5 \times 10^5 \\
0 & 0 & -500 \times 10^5
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
10 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

The process state variables 1, 2 and 3 correspond to \(T_0\), \(T_F\) and \(T_R\), respectively, and states 4 and 5 correspond to the \(O_2\) and \(CO\) concentrations in the furnace. The manipulated variables 1, 2 and 3 correspond to feed flow rate, \(F_{\text{feed}}\), fuel flow rate, \(F_{\text{fuel}}\), and furnace vent position, \(P_v\), respectively. The nominal value of the state and manipulated variables are \(s^{\text{nom}} = [300K; 375K; 500K; 4\%; 100\, \text{ppm}]\) and \(m^{\text{nom}} = [10000 \, \text{bbl/day}; 10 \, \text{bbl/day}; 0.1\%].\)

The disturbance is a white noise process with a mean of \(\bar{p} = 300K\) and a spectral density of \(\Sigma_p = 100\). The constraints on state variables and manipulated variables are: \(370K \leq T_F \leq 380K\), \(495K \leq T_R \leq 505K\), \(1\% \leq C_{CO} \leq 7\%\), \(70\, \text{ppm} \leq C_{CO} \leq 130\, \text{ppm}\), \(9900\, \text{bbl/day} \leq F_{\text{fuel}} \leq 10100\, \text{bbl/day}\), \(8\, \text{bbl/day} \leq F_{\text{fuel}} \leq 12\, \text{bbl/day}\), \(0.09\% \leq P_v \leq 0.11\%\). It is assumed that two types of sensors are available for each state variable. The data of the sensors is provided in Table 5 and generates a matrix \(C\) equal to \([c_1^T \ c_1^T \ c_2^T \ c_3^T \ c_3^T \ c_4^T \ c_4^T \ c_5^T \ c_5^T]^T\)

where \(c_i = \rho_i\).
Table 5: Sensor data for reactor furnace example.

<table>
<thead>
<tr>
<th>i</th>
<th>Type</th>
<th>Location</th>
<th>$\sigma^2_{v_i}$</th>
<th>$c_i^{(s)}$(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Temp</td>
<td>$c_1$</td>
<td>0.0625</td>
<td>300</td>
</tr>
<tr>
<td>2</td>
<td>Temp</td>
<td>$c_1$</td>
<td>0.25</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>Temp</td>
<td>$c_2$</td>
<td>0.0625</td>
<td>300</td>
</tr>
<tr>
<td>4</td>
<td>Temp</td>
<td>$c_2$</td>
<td>0.25</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>Temp</td>
<td>$c_3$</td>
<td>0.0625</td>
<td>300</td>
</tr>
<tr>
<td>6</td>
<td>Temp</td>
<td>$c_3$</td>
<td>0.25</td>
<td>200</td>
</tr>
<tr>
<td>7</td>
<td>$O_2$ Concentration</td>
<td>$c_4$</td>
<td>0.1</td>
<td>1800</td>
</tr>
<tr>
<td>8</td>
<td>$O_2$ Concentration</td>
<td>$c_4$</td>
<td>0.4</td>
<td>1200</td>
</tr>
<tr>
<td>9</td>
<td>CO Concentration</td>
<td>$c_5$</td>
<td>0.1</td>
<td>2400</td>
</tr>
<tr>
<td>10</td>
<td>CO Concentration</td>
<td>$c_5$</td>
<td>0.4</td>
<td>1600</td>
</tr>
</tbody>
</table>

The economic objective function (to be maximized) is:

$$g = 10F_{feed} - 30F_{fuel} - 0.1C_{CO} - \sum_{i=1}^{n} c_i^{(s)}\alpha_i$$

Both the branch and bound method and the GBD approach obtain the same optimal solution: two temperature sensors with variance of 0.0625 measuring the feed flow temperature and furnace temperature, with an optimal objective function value of $99,175. The GBD implementations achieves over 84% improvement in computation time, as indicated in Table 6.

Table 6: Comparison of computational efforts for Example 3.

<table>
<thead>
<tr>
<th>Method</th>
<th>Iterations</th>
<th>SDP Calls</th>
<th>Time (sec)</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Branch and Bound</td>
<td>78</td>
<td>156</td>
<td>152</td>
<td>-</td>
</tr>
<tr>
<td>GBD</td>
<td>14</td>
<td>14</td>
<td>23</td>
<td>84.9%</td>
</tr>
</tbody>
</table>

Example 4a: Consider the jacketed CSTR system of Figure 6, adapted from Ref. [20].
The process is described by the following material and energy balances:

\[
\dot{C}_A = \frac{F_0}{V} C_{A0} - \frac{F}{V} C_A - k_0 e^{-E/RT} C_A
\]

\[
\dot{T} = \frac{F_0}{V} T_0 - \frac{F}{V} T + \frac{(-\Delta H)}{\rho C_p} k_0 e^{-E/RT} C_A - \frac{UA(T - T_c)}{V \rho C_p}
\]

\[
\dot{T}_c = \frac{F_c}{V_j} (T_{c0} - T_c) + \frac{UA(T - T_c)}{V_j \rho_j C_{pj}}
\]

\[
\dot{V} = F_0 - F
\]

\[
\dot{P} = \frac{RT}{64 - V} (V k_0 e^{-E/RT} C_A - F_{vg})
\]

The state, manipulated and disturbance vectors are \( s = [C_A T T_c V P]^T \), \( m = [F F_c F_{vg}]^T \) and \( p = [F_0 C_{A0}]^T \). The manipulated inputs \( F, F_c \) and \( F_{vg} \) have nominal values of 40 \( ft^3/h \), 56 \( ft^3/h \) and 10.6 \( lb \cdot mol/h \), respectively. The disturbance inputs \( F_0 \) and \( C_{A0} \) have mean values of 40 \( ft^3/h \) and 0.5 \( lb \cdot mol/ft^3 \), respectively. The spectral density of \( F_0 \) and \( C_{A0} \) equal to 0.12 and 0.012, respectively. The operating constraints are as follows.

- 0.15 \( lb \cdot mol/ft^3 \) \( \leq C_A \) \( \leq 0.35 \( lb \cdot mol/ft^3 \), 595 \( ^\circ R \) \( \leq T \) \( \leq 650 \( ^\circ R \), 560 \( ^\circ R \) \( \leq T_c \) \( \leq 620 \( ^\circ R \),
- 40 \( ft^3 \) \( \leq V \) \( \leq 55 \( ft^3 \), 1600 \( lbf/ft^2 \) \( \leq P \) \( \leq 2600 \( lbf/ft^2 \), 0 \( \leq F \) \( \leq 80 \( ft^3/h \), 0 \( \leq F_c \) \( \leq 200 \( ft^3/h \), 0 \( \leq F_{vg} \) \( \leq 20.5 \( lb \cdot mol/h \). It is assumed that 1% precision sensors are available.
at each state, and each sensor has an annualized cost of $1000/yr.

The economic objective function (to be maximized) is defined as:

\[
g = 8760 \times \left[0.375(C_{A0} - C_A)F - 0.015F_c - 0.00225F_{vg}\right] - \sum_{i=1}^{n} c_i^{(s)} \alpha_i
\]

Given the process constraints and objective function, the OSSOP is found to be \( s^{OSSOP} = [0.15 \text{ lb-mol/ft}^3 622.5 \text{ °R} 610 \text{ °R} 40 \text{ ft}^3 2100 \text{ lbf/ft}^2], m^{OSSOP} = [40 \text{ ft}^3/\text{hr} 56.4 \text{ ft}^3/\text{hr} 14 \text{ lb-mol/hr}], \) with optimal objective function value of $43,724/yr. Linearizing the nonlinear dynamic model and objective function around the OSSOP, one can obtain the linear model

\[
\dot{x} = Ax + Bu + Gw, \quad g = g(s^{OSSOP}, m^{OSSOP}, \bar{p}) + d_g^T[x^T u^T w^T]^T,
\]

where \( d_g^T \) is the partial derivative of the objective function evaluated at the OSSOP, and \( A, B, G \) and \( d_g \) are given in the appendix.

The branch and bound method and the GBD approach are used to solve this sensor selection problem. Both approaches yield the same optimal solution: 1% sensors measuring \( C_A, V \) and \( P \), respectively, with an optimal objective function value of $33,369/yr. The branch and bound method required 660 iterations and a runtime of 2,125 s. Using the GBD approach, only 48 iterations and 71 s were required.

**Example 4b:** Continuing Example 4a, remove the assumption of a linear steady-state model and use the nonlinear steady-state model and nonlinear objective function directly. To make the nonlinear relaxed master problem easier to solve, the nonlinear model is converted to a dimensionless variable form. Then, the relaxed master problem (with mixed integer nonlinear constraints and nonlinear objective function) is solved using SCIP, [21]. In this case, the GBD approach required 75 iterations and 184 s. A summary of computational effort for the 3 cases of Example 4 is provided in Table 7.
Table 7: Comparison of computational efforts for Example 4.

<table>
<thead>
<tr>
<th>Method</th>
<th>Iterations</th>
<th>SDP Calls</th>
<th>Time (sec)</th>
<th>Percent Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Branch and Bound (Linear steady-state model)</td>
<td>660</td>
<td>1320</td>
<td>2125</td>
<td>-</td>
</tr>
<tr>
<td>GBD (Linear steady-state model)</td>
<td>48</td>
<td>48</td>
<td>71</td>
<td>96.7%</td>
</tr>
<tr>
<td>GBD (Nonlinear steady-state model)</td>
<td>75</td>
<td>75</td>
<td>184</td>
<td>91.3%</td>
</tr>
</tbody>
</table>

4. Conclusion

In this work, we have shown that the GBD algorithm can be a powerful tool for solving value-optimal SND problems. In addition to the improvement in computational efficiency, the proposed algorithm decomposed the problem in such a way that off-the-shelf solvers can be employed. Thus, significantly less effort will be required to implement as compared to previous procedures, which required an in-house MATLAB development of the branch-and-bound algorithm.

Acknowledgement

Both authors thank the National Science Foundation (CBET-1511925) for financial support.

References


Appendix

\[
A = \begin{bmatrix}
-3.3333 & -0.0136 & 0 & -0.0088 & 0 \\
1.8667 \times 10^3 & -5.1442 & 15 & 7.0000 & 0 \\
0 & 93.8067 & -108.457 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
4.8176 \times 10^3 & 28.0176 & 0 & 18.0661 & 0 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-1.0000 & 0 & 0 & 0 & 0 \\
-0.0037 & 0 & 0 & 0 & 13.2500 \\
-15.5631 & 0 & 0 & 0 & 0 \\
0 & -20.7858 & 0 & 0 & 0 \\
0 & 0 & -51.6176 & 0 & 0 \\
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
1.0000 & 0 \\
0.0125 & 1.0000 \\
13.2500 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
d_g^T = \begin{bmatrix}
-1.3140 \times 10^5 & 0 & 0 & 0 & 1.1497 \times 10^3 & -131.4000 & -19.7100 & 1.3140 \times 10^5 \\
\end{bmatrix}
\]