Constrained Infinite-Time Nonlinear Quadratic Optimal Control

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Outline

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Constrained Infinite-Time Nonlinear Quadratic-Optimal Control
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Infinite-time Nonlinear Quadratic Optimal Control Problem

\[
\min_u \int_0^\infty (x^T Q(x)x + u^T R(x)u) \, dt
\]

s.t. \( \dot{x} = f(x) + B(x)u \)

The solution can be found by solving the HJB equation

\[
\frac{\partial J^T}{\partial x} f(x) + x^T Q(x)x - \frac{1}{4} \frac{\partial J^T}{\partial x} B(x) R^{-1}(x) B^T(x) \frac{\partial J}{\partial x} = 0
\]

The optimal policy is given by \( u^*(x) = -\frac{1}{2} R^{-1}(x) B^T(x) \left( \frac{\partial J}{\partial x} \right)^T \)

The determination of \( J(x) \) is nearly impossible in all but the simplest cases.
Infinite-time Nonlinear Quadratic Optimal Control

\( J(x) \) can be written as: \( J(x) = x^T P(x) x \); where \( P(x) \) is symmetric.

In this case:

\[
\frac{\partial J}{\partial x} = 2 \left( P(x) + \frac{1}{2} x^T \frac{\partial P}{\partial x} \right) x
\]

Define

\[
\Pi(x) = P(x) + \frac{1}{2} x^T \frac{\partial P}{\partial x}
\]

Then the HJB becomes:

\[
x^T \left[ \Pi(x) A(x) + A^T(x) \Pi(x) + Q(x) - \Pi(x) B(x) R^{-1}(x) B^T(x) \Pi(x) \right] x = 0
\]

where \( A(x) \) s.t. \( f(x) = A(x) x \).
Assumptions

(H1) $R(x) > 0, \forall \ x \in \mathbb{R}^n$, and $Q(x) > 0, \forall \ x \in D \subset \mathbb{R}^n$

(H2) $A(x)$ and $B(x)$ are analytic matrix valued functions $\forall \ x \in D$

(H3) The pair $(A(x), B(x))$ is controllable (in the linear system sense) $\forall \ x \in D$

(H4) The gradients $\frac{\partial (A(x)x)}{\partial x}$ and $\frac{\partial b_i(x)}{\partial x}$ exist and are continuous and bounded $\forall \ x \in D$ (where $b_i(x)$ is the $i$th column of $B(x)$)
State Dependent Riccati Equation

If $\Pi(x)$ satisfies the SDRE:

$$
\Pi(x)A(x) + A^T\Pi(x) - \Pi(x)B(x)R^{-1}(x)B^T(x)\Pi(x) + Q(x) = 0
$$

then the HJB will appear to be satisfied. In this case, the optimal policy is

$$
u^*(x) = -R^{-1}(x)B^T(x)\Pi(x)x
$$

and the optimal value function is

$$J(x) = x^TP(x)x$$
Gradient of a Scalar Function

For the HJB to actually be satisfied

\[ 2x^T \Pi(x) \text{ must equal } \frac{\partial J}{\partial x} \]

The conditions for a vector function, \( v(x) \), to be a gradient are:

\[ \frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} ; \forall \ i, j \]

Assumption: (H5) \( A(x) \) is s.t. the solution to the SDRE, \( \Pi(x) \), satisfies:

\[ curl(x^T \Pi(x)) = 0 ; \forall \ x \in D \]
Existence of a $\Pi(x)$

The following guarantees the existence of a SDRE such that the curl condition is satisfied (Lu and Huang, 1996).

If there exists a solution to the HJB equation, $J(x)$, then there exists $A(x)$ s.t. the solution to the SDRE equation, $\Pi(x)$, satisfies

$$\frac{\partial J}{\partial x} = 2x^T \Pi(x).$$
Stability and Optimality

**Theorem:** (Manousiouthakis and Chmielewski, 1998)
Let (H1)-(H5) hold. Then the scalar function

\[ J(x) = x^T \left( \int_0^1 \lambda \Pi(\lambda x) d\lambda \right) x \]

satisfies the necessary conditions for optimality (HJB) for all \( x \in D \). Furthermore, the feedback policy

\[ u(x) = -R^{-1}(x)B^T(x)\Pi(x)x \]

is asymptotically stabilizing.

**Corollary:** Let (H1)-(H5) hold, and assume \( D = \mathbb{R}^n \).
Then the feedback policy is globally asymptotically stabilizing.
An Inverse Method

Consider $\Pi(x)$ and $A(x)$ to be design functions.

First we must guarantee

$$\Pi(x) > 0, \quad \text{curl}(x^T \Pi(x)) = 0, \quad A(x)x = f(x)$$

Then simply calculate $Q(x)$ from the SDRE.

$$Q(x) = \Pi(x)B(x)R^{-1}(x)B^T(x)\Pi(x) - A^T(x)\Pi(x) - \Pi(x)A(x)$$
An Inverse Method (continued)

Let \( Z_\alpha \) be defined as (a positively invariant set)

\[
Z_\alpha = \left\{ x \in \mathbb{R}^n | J(x) \leq \alpha \right\}; \quad J(x) = x^T \left[ \int_0^1 \lambda \Pi(\lambda x) d\lambda \right] x
\]

Define \( \alpha^* \) s.t.

\[
x^T Q(x)x + u^T R(x)u = x^T \left[ Q(x) + \Pi(x) B(x) R^{-1}(x) B^T(x) \Pi(x) \right] x
\]

\[
> 0 \quad \forall \; x \in Z_{\alpha^*}
\]

Then a sufficient condition closed-loop stability is that \( x(0) \in Z_{\alpha^*} \).
Example

Consider a CSTR Reactor with reaction $2A \rightleftharpoons B$.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-2k_a x_1^2 - (4k_a C_{Ass} + \frac{F}{V}) x_1 + 2k_b x_2 \\
k_a x_1^2 + 2k_a C_{Ass} x_1 - (\frac{F}{V} + k_b) x_2
\end{bmatrix} + \begin{bmatrix}
\frac{F}{V} \\
0
\end{bmatrix} u
\]

\[
x_1 = C_A - C_{Ass} \quad ; \quad x_2 = C_B - C_{Bss} \quad ; \quad u = C_{Ain} - C_{Ainss}
\]

\[
C_{Bss} = \frac{k_a}{\frac{F}{V} + k_b} C_{Ass}^2 = \frac{1}{2} (C_{Ainss} - C_{Ass})
\]
Example (continued)

The following SS design parameters are used (all in SI units)

\[ k_a = 0.05 \quad k_b = 0.01 \quad \frac{F}{V} = 0.02 \]

\[ C_{A_{inss}} = 10 \quad C_{A_{ss}} = 1.59 \quad C_{B_{ss}} = 4.21 \]
Example (continued)

Let $\Pi(x) = P_o$ where $P_o$ satisfies

$$A_o^T P_o + P_o A_o^T + Q_o - P_o B R^{-1} B^T P_o = 0$$

where $R = (V/F)^2$,

$$A_o = \begin{bmatrix} - (4k_a C_{Ass} + \frac{F}{V}) & \frac{2k_b}{V} \\ 2k_a C_{Ass} & -(\frac{F}{V} + k_b) \end{bmatrix} \quad \text{and} \quad Q_o = \begin{bmatrix} q_{11}^o & 0 \\ 0 & q_{22}^o \end{bmatrix}$$

Clearly this choice of $\Pi(x)$ satisfies the “Curl” condition.
Example (continued)

$Q(x)$ is calculated, via the SDRE, to be:

$$Q(x) = \begin{bmatrix} q_{11}^o + 2k_a x_1 (2p_{11} - p_{12}) & k_a x_1 (p_{22} - 2p_{12}) \\ k_a x_1 (p_{22} - 2p_{12}) & q_{22}^o \end{bmatrix}$$

Then $Q(x) + P_o B R^{-1} B^T P_o$ is positive definite iff

$$q_{11}^o + 2k_a x_1 (2p_{11} - p_{12}) + p_{11}^2 > 0$$

$$(q_{11}^o + 2k_a x_1 (2p_{11} - p_{12}) + p_{11}^2) q_{22}^o - (k_a x_1 (p_{22} - 2p_{12}) p_{11} p_{12})^2 > 0$$
Example (continued)

For \( q_{11}^o = 1 \), and \( q_{22}^o = 0.1 \)

\[
x_1 \in ( -10 \, , \, 397 ) \Rightarrow Q(x) + P_o B R^{-1} B^T P_o > 0
\]

The largest positively invariant set

\[
Z_\alpha = \{ \xi \in \mathbb{R}^2 | \xi^T P_o \xi \leq \alpha \}
\]

such that \( Q(x) + P_o B R^{-1} B^T P_o > 0 \ \forall \ x \in Z_\alpha \), corresponds to

\[
\alpha = 71.68
\]
Example 1: Closed-Loop Stability Region

\[ Q(x) + P_o B R^{-1} B^T P_o > 0 \]
Example 1: Closed-Loop Simulations

[Diagram showing concentration over time for Nominal Inputs and Optimal Control, with labels CB and CA for concentration values.]
Constrained ITNQOC Problem

\[ \Phi(\xi) = \inf_u \left\{ \int_0^\infty [x^T Q x + \rho^T(u) R \rho(u)] \, dt \right\} \]

s.t. \[ \dot{x} = A(x) x + B(x) \rho(u); \quad x(0) = \xi \]

where \( \rho(\cdot) : \mathbb{R}^M \rightarrow U \), is defined as

\[ \rho(u) = \arg \min_{\mu \in U} ||u - \mu|| \]

Further Assumptions:

(H6) \( U \) is convex and contains the origin in its interior

(H7) \( \xi \in X_o = \{ \xi \in \mathbb{R}^n | \exists u \text{ s.t. } \Phi(\xi) < \infty \} \)
Consider the following family of finite-time optimal control problems.

\[ \Phi_T(\xi) = \inf_u \left\{ \int_0^T \left[ x^T Q x + \rho^T(u) R \rho(u) \right] dt + J(x(T)) \right\} \]

s.t. \( \dot{x} = A(x)x + B(x)\rho(u) \); \( x(0) = \xi \)

where

\[ J(x) = x^T \left[ \int_0^1 \lambda \Pi(\lambda x) d\lambda \right] x \]
Equivalence of Constrained and Unconstrained Problems

Define

\[ \bar{X} = \{ \xi \in \mathbb{R}^n \mid K(\xi)\xi \in U \} \]

\[ O_\infty \equiv \{ \xi \in \mathbb{R}^n \mid x(t) \in \bar{X} \ \forall \ t > 0 \} \]

where \( K(x) \) is the unconstrained optimal feedback gain.

**Lemma:** Let \( K(x) \) be bounded for all \( x \) in a neighborhood of the origin. Then

\[ 0 \in \text{int}\{U\} \implies 0 \in \text{int}\{O_\infty\} \]
Finite-Time Solution to Infinite-Time Problem

**Lemma:** Let (H1-H7) hold and assume the solution to $\Phi_T$ is unique. Then

(i) $\Phi_\infty \doteq \lim_{T \to \infty} \Phi_T$ exists for all $\xi \in X_o$
(ii) $\Phi_T = \Phi_\infty \iff \Phi_T = \Phi_{T+\tau}; \ \forall \tau \geq 0 \iff u_T^* = u_{T+\tau}^*; \ \forall \tau \geq 0 \iff x_T^*(T) \in O_\infty$
(iii) If $\exists T > 0$ such that $x_T^*(T) \in O_\infty$ then $\Phi_T = \Phi$

**Theorem:** Let (H1-H7) hold and assume the solution to $\Phi_T$ is unique. Then

$$\forall \xi \in X_o \ \exists T \text{ s.t. } x_T^*(T) \in O_\infty$$
Equivalence of Initial Condition Sets

Define the set of constrained stabilizable initial conditions as

\[ X_{\text{max}} = \left\{ \xi \in \mathbb{R}^n \mid \exists u \text{ s.t. } \lim_{t \to \infty} \|x(t)\| = 0 \right\} \]

**Theorem:** Let (H1-H7) hold and assume the solution to \( \Phi_T \) is unique. Then

\[ X_{\text{max}} = X_0 \]
Example 2

Consider the CSTR of Example 1 with $k_b = 0$

$$
\dot{x}_1 = -2k_a x_1^2 - \left( 4k_a C_{Ass} + \frac{F}{V} \right) x_1 + \frac{F}{V} u
$$

with objective function ($r = \left( \frac{F}{V} \right)^2$)

$$
\int_0^\infty \left\{ x_1(t)^2 + ru(t)^2 \right\} dt
$$

and constraints

$$
u(t) + C_{Ass} \in [0, 100] \quad \forall \quad t \geq 0 \quad \Rightarrow \quad X_{max} = (-3.89, \infty)
$$
Example 2 (continued)

The resulting SDRE is:

\[ 2 \left( -2k_{a}x_{1} - 2k_{a}C_{Ass} \right) \pi(x_{1}) + 1 - \pi^{2}(x_{1}) = 0 \]

Which has solution

\[ \pi(x_{1}) = 2 \left( k_{a}x_{1} + k_{a}C_{Ass} + \sqrt{(k_{a}x_{1} + k_{a}C_{Ass})^{2} + 1} \right) \]
Example 2 (continued)

The Unconstrained Optimal Control Policy is

\[ u(x) = -\frac{V}{F} \left( k_a x_1 + k_a C_{Ass} + \sqrt{(k_a x_1 + k_a C_{Ass})^2 + 1} \right) x_1 \]

and

\[ O_\infty = (-0.74, \infty) \]

Finally, the Closed-Loop System is

\[ \dot{x} = -x_1 \sqrt{(k_a x_1 + k_a C_{Ass})^2 + 1} \]
Example 2: Optimal State Trajectories

- Red dashed line: No Constraints
- Blue solid line: Constrained Input
- Black dotted line: Nominal Input
Example 2: Optimal Input Policies

The graph illustrates two input policies:
- **No Constraints** (dashed red line)
- **Constrained Input** (solid blue line)

The graph shows the concentration of component A (CA) over time. The constrained input policy maintains a lower concentration over time compared to the unconstrained policy, demonstrating a more controlled and stable system behavior.
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