Constrained Infinite-time Linear Quadratic Games

Don Chmielewski

Vasilios Manousiouthakis

UCLA Chemical Engineering

Abstract: In this work, we explicitly consider additive disturbances in the development of a constrained infinite-time minimax optimal control policy. Direct calculation of the closed-loop solution to the constrained minmax game (i.e. through the Isaacs’ equation) is typically intractable due to the curse of dimensionality. We consider the open-loop information game, with the intent of using its solution in a real-time feedback implementation.
**Project Objectives**

- **Motivation:** Improve Process Performance in Disturbance Input Case
- **Goal:** Provide Computationally Feasible Solution Methodology
Closed-Loop Problem Formulation

\[
\inf_{\mu: \mathbb{R}^n \to U \subset \mathbb{R}^m} \sup_{\nu: \mathbb{R}^n \to W \subset \mathbb{R}^q} \left\{ \sum_{k=0}^{\infty} H(x_k) \right\} \quad \text{s.t.} \quad x_{k+1} = Ax_k + B\mu(x_k) + G\nu(x_k)
\]

\[
H(x) = \begin{cases} 
  x^TQx + \mu^T(x)\mu(x) - \gamma^2\nu^T(x)\nu(x) & \text{if } x \in X_{\text{max}} \\
  +\infty & \text{if } x \notin X_{\text{max}} 
\end{cases}
\]

where

- \( X_{\text{max}} = \{ x_0 \in X \mid \exists \{ u_k \}_{k=0}^{\infty} \in U \text{ s.t. } \{ x_k \}_{k=1}^{\infty} \in X \text{ and } x_k \to 0 \} \) is the constrained stabilizability set for the disturbance free problem.

- The Optimization Variables \( \mu(\cdot) \) and \( \nu(\cdot) \) are Nonlinear Operators from the State to the Inputs.

- \( \gamma > 0 \) and \( Q > 0 \) are fixed constants and the Sets \( X, U, \) and \( W \) are convex, compact and contain the origin in their interiors.
Unconstrained Solution (Basar and Bernhard, 1991)

\[
\mu^*(x_k) = -B^T P \Lambda^{-1} A x_k \quad \text{and} \quad \nu^*(x_k) = \gamma^{-2} G^T P \Lambda^{-1} A x_k
\]

where

\[
P = Q + A^T P \Lambda^{-1} A \quad \text{and} \quad \Lambda = I + \left( B B^T - \gamma^{-2} G G^T \right) P
\]

This saddle-point solution is guaranteed to exist if \((\gamma^2 I - G^T P G) > 0\)

The value of the game is

\[
\Psi^*_0(x_0) = x_0^T P x_0
\]

The worst case closed-loop response is

\[
x_{k+1} = \Lambda^{-1} A x_k
\]
Constrained Open-Loop Problem Formulation

\[
\Phi^*(x_0) = \sup_{\vec{w} \in \mathcal{W}} \Phi(\vec{w})
\]

where

\[
\Phi(\vec{w}) = \inf_{\vec{u} \in \mathcal{U}} \inf_{\vec{x} \in \mathcal{X}} H(\vec{x}, \vec{u}, \vec{w}) \quad \text{s.t.} \quad x_{k+1} = Ax_k + Bu_k + Gw_k
\]

\[
H(\vec{x}, \vec{u}, \vec{w}) = \begin{cases} 
\sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T u_k - \gamma^2 w_k^T w_k & \text{if } x_k \in X_{\text{max}} \quad \forall \ k \geq 0 \\
\infty & \text{otherwise}
\end{cases}
\]

where

\[
\mathcal{W} = \{ \vec{w} \in \ell_2^g \mid w_k \in W, \forall \ k \geq 0 \}
\]

\[
\mathcal{U} = \{ \vec{u} \in \ell_2^p \mid u_k \in U, \forall \ k \geq 0 \}
\]

\[
\mathcal{X} = \{ \vec{x} \in \ell_2^n \mid x_k \in X, \forall \ k \geq 0 \}
\]
Smoothness of the Objective Functions

Open-Loop Problem: If the initial condition is contained in

\[
X^{(ol)}_{\max} = \{ x_0 \in \mathbb{R}^n \mid \forall \bar{w} \in \mathcal{W}, \ \exists \bar{u}(\bar{w}) \in \mathcal{U} \text{ s.t. } x_k \in X_{\max} \ \forall \ k \geq 0 \}
\]

then

\[
H(\bar{x}, \bar{u}, \bar{w}) = \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T u_k - \gamma^2 w_k^T w_k
\]

Closed-Loop Problem: If the initial condition is contained in

\[
X^{(cl)}_{\max} = \{ x_0 \in \mathbb{R}^n \mid \forall \bar{w} \in \mathcal{W}, \ \exists \mu(\cdot) : \mathbb{R}^n \to \mathcal{U} \text{ s.t. } x_k \in X_{\max} \ \forall \ k \geq 0 \}
\]

then

\[
H(x) = x^T Q x + \mu^T(x) \mu(x) - \gamma^2 \nu^T(x) \nu(x)
\]
Finite Dimensional Solution to Open-Loop Problem

Consider the Finite-time Game:

$$\Phi^*_N(x_0) = \max_{\bar{w} \in \mathcal{W}} \left\{ \min_{\bar{u} \in \mathcal{U}, \bar{x} \in \mathcal{X}} \right. $$

$$\left. \Sigma_{k=0}^{N-1} \left( x_k^T Q x_k + u_k^T u_k - \gamma^2 w_k^T w_k \right) + x_N^T P x_N \right\}$$

$$\text{s.t. } x_{k+1} = A x_k + B u_k + G w_k$$

If $x_0 \in X_{\text{max}}^{(ol)}$ and the solution to $\Phi^*_N$ is such that

$$x^*_N \in \mathcal{O}_\infty \overset{\hat{=}}{=} \left\{ x_0 \text{ s.t. open-loop solutions to the constrained and unconstrained problems coincide} \right\}$$

then the finite-time game will provide the solution to the infinite-time game.
Example

Consider the One State System:

\[ x_{k+1} = 0.9x_k + u_k + w_k \]

With Constraints:

\[ |u_k| \leq 0.3 \text{ , and } |w_k| \leq 0.05 \]

And Design Parameters:

\[ Q = 1 \text{ , and } \gamma^2 = \sqrt{3} \]
Solution to the Closed Loop Problem

The Optimal Closed Loop Policies are found by solving the Isaacs’ Equation for $\Psi: X_{\text{max}}^{(cl)}(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}$:

$$\Psi(x) = \min_{u \in U} \max_{w \in W} \left\{ \begin{array}{ll}
( x^T Q x + u^T u - \gamma^2 w^T w + \Psi(Ax + Bu + Gw) ) & \text{if } Ax + Bu + Gw \in X_{\text{max}} \\
+\infty & \text{otherwise}
\end{array} \right\}$$

$\exists \gamma > 0$ such that the Value Function; $\Phi^*: X_{\text{max}}^{(cl)} \rightarrow \mathbb{R}$, will satisfy the Constrained Closed-Loop Infinite-time Isaacs’ Equation.