A Globally Optimal, Dynamic Based, Operating Point Selection Scheme for MPC

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Abstract: We propose a new formulation of the stochastically based minimally backed-off operating point (MBOP) selection problem. This scheme aims to combine the steady-state notions of profit with the dynamic, constraint observing notions of MPC design and tuning. The proposed formulation has a convex / reverse-convex form, and is readily solved globally via branch and bound. The formulation is trivially extended to the partial state information and discrete-time cases.

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Real-Time Optimization

Feasible Steady-State Operating Point

Constraint Polytope of Feasible Operating Points

Optimal Steady-State Operating Point
Minimally Backed-off Operating Point (MBOP) Selection

**Goal:** Bring the Backed-off Point as close as possible to the Optimal Steady-State.

**Constraint:** Do not allow the Expected Dynamic Operating Region outside the Constraint Polytope.

**Steady-State Operating Line:** Backed-off Points further limited by the Steady-State model.

**Controller Tuning:** Different tuning values will change the Size and Shape of the Expected Dynamic Operating Region.
**Illustrative Example**

**System Model:**
\[
\begin{bmatrix}
\dot{r} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-2 & -3
\end{bmatrix}
\begin{bmatrix}
r \\
v
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} f +
\begin{bmatrix}
0 \\
1
\end{bmatrix} w
\]

where \( r \) is the mass position, \( v \) is the velocity, \( f \) is the input force (MV) and \( w \) is the disturbance force

**System Constraints:**
\[-1 \leq r \leq 1 \quad \text{and} \quad 0 \leq f \leq 16\]

**Design Objectives:**

**Steady-State:** Put the average mass position as close as possible to the upper bound

**Dynamic:** In the face of disturbances, do not allow the mass position trajectory to extend beyond the upper bound.
Example Problem Formulation

\[
\begin{align*}
\min & \quad \tilde{r}, \tilde{f}, x_{\min}, x_{\max}, u_{\min}, u_{\max}, \zeta_x, \zeta_u, L, \Sigma_x \geq 0 \\
\text{s.t.} & \quad \tilde{f} = 3\tilde{r}, \\
& \quad -2 \leq \tilde{r} \leq 0, \quad -12.8 \leq \tilde{f} \leq 2.2 \\
& \quad x_{\min} = \tilde{r} + 2, \quad x_{\max} = \tilde{r}, \\
& \quad u_{\min} = \tilde{f} + 12.8, \quad u_{\max} = \tilde{f} - 2.2, \\
& \quad \zeta_x < (x_{\min})^2, \quad \zeta_x < (x_{\max})^2, \\
& \quad \zeta_u < (u_{\min})^2, \quad \zeta_u < (u_{\max})^2, \\
& \quad \zeta_x = [1 \ 0][\Sigma_x [1 \ 0]^T, \quad \zeta_u = L[\Sigma_x L]^T, \\
& \quad (A + BL)[\Sigma_x + \Sigma_x (A + BL)^T + G\Sigma_w G^T = 0
\end{align*}
\]

\(\tilde{r}\) and \(\tilde{f}\) are deviation variables w.r.t. OSSOP. 
\(x_{\min}\) is distance from BOP to constraint. 
\(2\zeta_x\) is the EDOR height. 
\(\zeta_x < (x_{\min})^2\) guarantees EDOR within constraints.
**Numeric Solutions**

**FSI Case:** Full State Information.  
Controller is \( u(t) = Lx(t) \)

**PSI Case:** Partial Information: 1 Velocity Sensor.  
Controller is \( u(t) = L\hat{x}(t) \)  
where \( \hat{x}(t) \) is from a state estimator.

**Case A:** Same as FSI Case.

**Case B:** Same as Case A, but max force changed from 15 to 18.

**Case C:** Same as Case A, but min force changed from 0 to 9.5.
General MBOP Formulation

Dynamic System in Actual Variables:

\[
\dot{s} = As + Bm + Gp, \quad z = Dx s + Du m + Dw p, \quad d_{\min} \leq z_{ss} \leq d_{\max}
\]

Controlled System in Deviation Variables:

\[
\dot{x} = Ax + Bu + Gw, \quad u = Lx, \quad \text{Size of } w \text{ given by } \Sigma_w.
\]

General Problem Formulation:

\[
\min_{\tilde{s}_{ss}, \tilde{m}_{ss}, \tilde{z}_{ss}, \zeta, L, \Sigma_x \geq 0} \quad d_s^T \tilde{s}_{ss} + d_m^T \tilde{m}_{ss}
\]

s.t. \quad 0 = A\tilde{s}_{ss} + B\tilde{m}_{ss}, \quad \tilde{z}_{ss} = Dx \tilde{s}_{ss} + Du \tilde{m}_{ss}, \quad \tilde{d}_{\min} \leq \tilde{z}_{ss} \leq \tilde{d}_{\max}

\[
\zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\min,i})^2, \quad \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\max,i})^2, \quad i = 1 \cdots n_z
\]

\[
\zeta_i = \phi_i [(D_x + Du L)\Sigma_x (D_x + Du L)^T + Dw \Sigma_w D_w^T] \phi_i^T, \quad i = 1 \cdots n_z
\]

\[
0 = (A + BL)\Sigma_x + \Sigma_x (A + BL)^T + G\Sigma_w G^T
\]
Constraint Convexification Theorem

\[ \exists \text{ stabilizing } L, \Sigma_x \geq 0, \text{ and } \zeta_i, \quad i = 1 \cdots n_z \]

s.t. \( (A + BL)\Sigma_x + \Sigma_x (A + BL)^T + G\Sigma_w G^T = 0, \quad \zeta_i < \bar{z}_i^2, \quad i = 1 \cdots n_z \]

and \( \zeta_i = \phi_i [(D_x + D_u L)\Sigma_x (D_x + D_u L)^T + D_w \sum_w D_w^T] \phi_i^T, \quad i = 1 \cdots n_z \)

if and only if

\[ \exists \quad L, X > 0 \quad \text{and} \quad \zeta_i, \quad i = 1 \cdots n_z \]

s.t. \( (AX + BY) + (AX + BY)^T + G\sum_w G^T < 0, \quad \zeta_i < \bar{z}_i^2, \quad i = 1 \cdots n_z \]

and \[ \begin{bmatrix} \zeta_i - \phi_i D_w \sum_w D_w^T \phi_i^T & \phi_i (D_x X + D_u Y) \\ (D_x X + D_u Y)^T \phi_i^T & X \end{bmatrix} > 0 \quad , \quad i = 1 \cdots n_z \]
Reverse-Convex Constraints

Reverse-Convex Constraints required to Guarantee EDOR within the Polytope:

\[ \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\text{min},i})^2 \]

and

\[ \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\text{max},i})^2 \]

Branch and Bound Algorithm used to find Globally Optimal Solutions
Reactor Furnace Example

Manipulated Variables:
- Reactant Feed Rate
- Fuel Feed Rate
- Vent Position

State Variables:
- Reactor Temperature
- Furnace Temperature
- Furnace $O_2$
- Furnace CO

Disturbance Input:
- Feed Temperature
**Numeric Solutions**

Profit = \(-0.01 \, C_{O_2} + 10 \, F_{in} - 30 \, F_{fuel}\)

PSI case uses sensor at \(T_R\)
Comparison of Profits

**OSSOP:**
Profit = $100,704

**Case A:** Same as FSI Case.
Profit = $100,698

**Case B:** Same as Case A, but fuel feed bounds changed to 10 ± 0.25.
Profit = $100,449

**Case C:** Same as Case A, but O2 concentration bound changed to 4%.
Profit = $100,599