Sensor Network Design using the MBOP Notion of Profit

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Outline

• Review of Covariance Analysis and Controller Design
• Covariance Based Sensor Selection
• Profit Based Operating Point Selection
• Profit Based Sensor Selection
Mass-Spring-Damper Example

Position of Mass: $r(t)$

External Force: $w(t)$
Impact of Disturbance Force
Impact of Disturbance Force

$w(t)$

$r(t)$

$v(t)$
Measure of Variability

\[ w(t) \]

\[ r(t) \]

\[ v(t) \]
Dynamic Operating Region

Trajectory of Position and Velocity
Expected Dynamic Operating Region (EDOR)

\[ r \]
\[ \sigma_r \]
\[ \sigma_v \]

\[ v \]
Expected Dynamic Operating Region (EDOR)

Steady-State Operating Point

\[ \sigma_r \]

\[ \sigma_v \]

Expected Dynamic Operating Region (EDOR)
Covariance Analysis

\[ w(t) \rightarrow \text{Plant} \rightarrow x(t) \]

**Process Model:**

\[ \dot{x}(t) = Ax(t) + Gw(t) \]

**Steady State Covariance:**

\[ A \Sigma_x + \Sigma_x A^T + G \Sigma_w G^T = 0 \]
**Expected Dynamic Operating Region (EDOR)**

EDOR defined by:

\[
\sum_x = \begin{bmatrix}
\sigma_{11}^2 & \sigma_{12}^2 \\
\sigma_{21}^2 & \sigma_{22}^2
\end{bmatrix}
\]
Closed-Loop Covariance Analysis

**Process Model:**
\[ \dot{x} = Ax + Bu + Gw \]

**Controller:**
\[ u(t) = Lx(t) \]

**Covariance of the State:**
\[ (A + BL) \Sigma_x + \Sigma_x (A + BL)^T + G \Sigma_w G^T = 0 \]

**Covariance of the Input:**
\[ \Sigma_u = L \Sigma_x L^T \]
Closed-Loop EDOR

defined by:

$$\Sigma_{xu} = \begin{bmatrix} \sigma_x^2 & \sigma_{xu}^2 \\ \sigma_{ux}^2 & \sigma_u^2 \end{bmatrix}$$
Closed-Loop EDOR

Closed-Loop EDOR of different controllers

\[ u = L_1 x \]

\[ u = L_2 x \]
Minimum Variance Design

Minimum Variance Problem:

\[
\text{Min}_{L} \sum_{i} \alpha_{i}(\phi_{i}\Sigma_{x}\phi_{i}^{T}) + \sum_{i} \beta_{i}(\phi_{i}\Sigma_{u}\phi_{i}^{T})
\]

s.t. \((A + BL)\Sigma_{x} + \Sigma_{x}(A + BL)^{T} + G\Sigma_{w}G^{T} = 0\)

\[
\Sigma_{u} = L\Sigma_{x}L^{T}
\]

\[
\phi_{i} = [0 \ 0 \ \ldots \ 1 \ \ldots \ 0 \ 0]
\]

\[i^{th}\] column
Minimum Closed-Loop EDOR
Constrained Minimum Variance Problem:

\[
\text{Min } \sum_{L} \alpha_i (\phi_i \Sigma_x \phi_i^T) + \sum_i \beta_i (\phi_i \Sigma_u \phi_i^T)
\]

s.t. \((A + BL)\Sigma_x + \Sigma_x (A + BL)^T + G\Sigma_w G^T = 0\)

\[
\Sigma_u = L\Sigma_x L^T, \quad \Sigma_x < \Sigma_x, \Sigma_u < \Sigma_u
\]

\[
\phi_i = \begin{bmatrix} 0 & 0 & \ldots & 1 & \ldots & 0 & 0 \end{bmatrix}
\]

\[\uparrow i^{th} \text{ column}\]
Constrained Minimum Variance Problem:

\[
\text{Min} \sum_{i} \alpha_i (\phi_i \Sigma_x \phi_i^T) + \sum_{i} \beta_i (\phi_i \Sigma_u \phi_i^T)
\]

\[
s.t. \quad (A + BL)\Sigma_x + \Sigma_x (A + BL)^T + G \Sigma_w G^T = 0
\]

\[
\Sigma_u = L \Sigma_x L^T, \quad \Sigma_x < \Sigma_x, \Sigma_u < \Sigma_u
\]

\[
\phi_i = [0 \ 0 \ \ldots \ 1 \ \ldots \ 0 \ 0]
\]

\[\uparrow \ i^{th} \text{ column} \]
Constrained Closed-Loop EDOR

\[ \Sigma_x < \bar{\Sigma}_x \text{ and } \Sigma_u < \bar{\Sigma}_u \]

Steady-State Operating Point
Constrained Closed-Loop EDOR

Constraints
\((\Sigma_x < \bar{\Sigma}_x \text{ and } \Sigma_u < \bar{\Sigma}_u)\)

Steady-State Operating Point

\[ \mathbf{x} \]

\[ \mathbf{u} \]
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- Review of Covariance Analysis and Controller Design
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- Profit Based Sensor Selection
State Estimation

Process Model: \[ \dot{x} = Ax + Bu + Gw \quad w \sim \Sigma_w \]

Measurements: \[ y = Cx + v \quad v \sim \Sigma_v \]

State Estimator: \[ \hat{x} = A\hat{x} + Bu + K(y - C\hat{x}) \]
**Estimation Error Covariance**

Define Estimation Error: \( e(t) = x(t) - \hat{x}(t) \)

Error System:

\[
\dot{e} = \left( A - \Sigma_e C^T \Sigma_v^{-1} C \right) e + G w - \Sigma_e C^T \Sigma_v^{-1} v
\]

Covariance of the Error:

\[
A \Sigma_e + \Sigma_e A^T + G \Sigma_w G^T - \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e = 0
\]
EDOR of Estimation Error

The diagram illustrates the estimation error with respect to the actual value $x_1$ and the estimated value $\hat{x}_1$, as well as the error with respect to $x_2$ and $\hat{x}_2$. The shaded area represents the Estimator EDOR.
Sensor Selection: Formulation 1
(Minimum Error Covariance Formulation)

Yu and Seinfeld (1973) propose:

\[
\begin{align*}
\text{Min} & \quad \text{Tr}\{\Sigma_e}\nonumber \\
\text{subject to} & \quad A\Sigma_e + \Sigma_e A^T + G\Sigma_w G^T 
\end{align*}
\]

\[
- \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e = 0
\]

Number of Sensors = Constant
Minimal Estimation Error EDOR

$\hat{x}_1 - x_1$

Poor Sensor Array

$\hat{x}_2 - x_2$

Optimal Sensor Array
Chmielewski et al. (2002) propose:

\[
\text{Min} \quad \left\{ \text{Cost of Sensor Array} \right\}
\]

\[
\text{s.t.} \quad A \Sigma_e + \Sigma_e A^T + G \Sigma_w G^T - \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e = 0
\]

\[
\Sigma_e \leq \bar{\Sigma}_e
\]

Variable Number of Sensors
Minimum Array Cost EDOR

Too Costly

Optimal

Constraints \( \sum e \leq \bar{\Sigma}_e \)

\[ x_1 - \hat{x}_1 \]

\[ x_2 - \hat{x}_2 \]
Sensor Selection: Formulation 3
(Closed-Loop Minimum Cost Formulation)

Peng and Chmielewski (2004) propose:

\[
\text{Min} \quad \{\text{Cost of Sensor Array}\}
\]

\[
\text{s.t.}
\]

\[
A\Sigma_x + \Sigma_x A^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^TB^T + G\Sigma_w G^T = 0
\]

\[
A\Sigma_e + \Sigma_e A^T + G\Sigma_w G^T - \Sigma_e C^T \Sigma_v^{-1} C\Sigma_e = 0
\]

\[
\Sigma_x \leq \overline{\Sigma_x}, \quad L(\Sigma_x - \Sigma_e)L^T = \Sigma_u \leq \overline{\Sigma_u}
\]
Constrained Closed-Loop EDOR

\[
\begin{align*}
\sum_x & \leq \bar{\sum}_x, \\
\sum_u & \leq \bar{\sum}_u
\end{align*}
\]
Outline

- Review of Covariance Analysis and Controller Design
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- Profit Based Operating Point Selection
- Profit Based Sensor Selection
Steady-State Operating Region

Steady-State Operating Point

CV’s

Constraints

MV’s
Real-Time Optimization

Steady-State Operating Point

Constraints

Optimal Steady-State Operating Point (OSSOP)

CV’s

MV’s
Minimally Backed-off Operating Point (MBOP)

Backed-off Operating Point (BOP)

Minimally Backed-off Operating Point (MBOP)

Optimal Steady-State Operating Point (OSSOP)

EDOR

CV’s

MV’s
Controller is fixed \iff\ EDORs have fixed sizes and shapes
Peng et al. (2004) propose:

Variable Controller  $\Rightarrow$ EDORs have variable sizes and shapes
MBOP Selection Formulation 2  
(Problem Formulation)

\[
\min \quad d^T_s \tilde{s}_{ss} + d^T_m \tilde{m}_{ss} \\
\tilde{s}_{ss}, \tilde{m}_{ss}, \tilde{z}_{ss}, \zeta, \Sigma_x \geq 0
\]

s.t.  
\[
0 = A \tilde{s}_{ss} + B \tilde{m}_{ss}; \quad \tilde{z}_{ss} = D_x \tilde{s}_{ss} + D_u \tilde{m}_{ss}, \quad \tilde{d}_{min} \leq \tilde{z}_{ss} \leq \tilde{d}_{max}
\]
\[
\zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{min,i})^2, \quad \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{max,i})^2, \quad i = 1 \ldots n_z
\]
\[
\zeta_i = \phi_i[(D_x + D_u L)(\Sigma_x - \Sigma_e)(D_x + D_u L)^T \\
+ D_x \Sigma_e D_x^T + D_w \Sigma_w D_w^T]\phi_i^T, \quad i = 1 \ldots n_z
\]
\[
0 = A\Sigma_x + \Sigma_x A^T + G\Sigma_w G^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^TB^T
\]
\[
0 = A\Sigma_e + \Sigma_e A^T - \Sigma_e C^T \Sigma_v^{-1} C\Sigma_e + G\Sigma_w G^T
\]
MBOP Selection Formulation 2  
(Problem Formulation)

\[ \min_{\tilde{s}_{ss}, \tilde{m}_{ss}, \tilde{z}_{ss}, \zeta, \Sigma_x \geq 0} \quad d_s^T \tilde{s}_{ss} + d_m^T \tilde{m}_{ss} \]

\[ s.t. \]  
\[ 0 = A \tilde{s}_{ss} + B \tilde{m}_{ss} ; \quad \tilde{z}_{ss} = D_x \tilde{s}_{ss} + D_u \tilde{m}_{ss} , \quad \tilde{d}_{\min} \leq \tilde{z}_{ss} \leq \tilde{d}_{\max} \]

\[ \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\min,i})^2 , \quad \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\max,i})^2 , \quad i = 1 \cdots n_z \]

\[ \zeta_i = \phi_i [(D_x + D_u L)(\Sigma_x - \Sigma_e)(D_x + D_u L)^T + D_x \Sigma_e D_x^T + D_w \Sigma_w D_w^T] \phi_i^T , \quad i = 1 \cdots n_z \]

\[ 0 = A \Sigma_x + \Sigma_x A^T + G \Sigma_w G^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^T B^T \]

\[ 0 = A \Sigma_e + \Sigma_e A^T - \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e + G \Sigma_w G^T \]
\[ \begin{align*} 
\min & \quad d_s^T \tilde{s}_{ss} + d_m^T \tilde{m}_{ss} \\
\tilde{s}_{ss}, \tilde{m}_{ss}, \tilde{z}_{ss}, & \quad \zeta, \Sigma_x \geq 0 \\
\end{align*} \]

s.t. \[ 0 = A\tilde{s}_{ss} + B\tilde{m}_{ss}; \quad \tilde{z}_{ss} = D_x \tilde{s}_{ss} + D_u \tilde{m}_{ss}, \quad \tilde{d}_{\min} \leq \tilde{z}_{ss} \leq \tilde{d}_{\max} \]

\[ \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\min,i})^2, \quad \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\max,i})^2, \quad i = 1 \ldots n_z \]

\[ \zeta_i = \phi_i [(D_x + D_u L)(\Sigma_x - \Sigma_e)(D_x + D_u L)^T \]
\[ + D_x \Sigma_x D_x^T + D_w \Sigma_w D_w^T ]\phi_i^T, \quad i = 1 \ldots n_z \]

\[ 0 = A\Sigma_x + \Sigma_x A^T + G\Sigma_w G^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^TB^T \]

\[ 0 = A\Sigma_e + \Sigma_e A^T - \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e + G\Sigma_w G^T \]
MBOP Selection Formulation 2
(Problem Formulation)

\[
\begin{align*}
\min & \quad d_s^T \tilde{s}_{ss} + d_m^T \tilde{m}_{ss} \\
\text{s.t.} & \quad 0 = A\tilde{s}_{ss} + B\tilde{m}_{ss}; \quad \tilde{z}_{ss} = D_x \tilde{s}_{ss} + D_u \tilde{m}_{ss}, \quad \tilde{d}_{\min} \leq \tilde{z}_{ss} \leq \tilde{d}_{\max} \\
& \quad \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\min,i})^2, \quad \zeta_i < (\tilde{z}_{ss,i} - \tilde{d}_{\max,i})^2, \quad i = 1 \ldots n_z
\end{align*}
\]

\[
\zeta_i = \phi_i [(D_x + D_u L)(\Sigma_x - \Sigma_e)(D_x + D_u L)^T \\
+ D_x \Sigma_e D_x^T + D_w \Sigma_w D_w^T] \phi_i^T, \quad i = 1 \ldots n_z
\]

\[
0 = A\Sigma_x + \Sigma_x A^T + G\Sigma_w G^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^T B^T
\]

\[
0 = A\Sigma_e + \Sigma_e A^T - \Sigma_e C^T \Sigma_v^{-1} C\Sigma_e + G\Sigma_w G^T
\]
Simple Example of MBOP Selection

Steady-State Objective:
Maximize the Mass Position

\[
\max \{ r_{ss} \} \\
\text{s.t.} \quad r_{\text{min}} \leq r_{ss} \leq r_{\text{max}} \\
\quad f_{\text{min}} \leq f_{ss} \leq f_{\text{max}} \\
\quad f_{ss} = 3r_{ss}
\]

The force \( f \) is the manipulated variable
The force \( w \) is the disturbance variable with size \( \sum w \)
Mass-Spring-Damper Example

Nominal Operation:

OSSOP

Backed-off Operating Point (BOP)

Upper Bound on Position

EDOR
Mass-Spring-Damper Example

Improved Operation:

- **OSSOP**
- Backed-off Operating Point (BOP)
- Upper Bound on Position

**EDOR**
Mass-Spring-Damper Example

Optimal Operation:

OSSOP

Minimally Backed-off Operating Point (MBOP)

Upper Bound on Position

EDOR
Mass-Spring-Damper Example

- OSSOP
- Steady-State Line
- Constraints

Diagram showing the relationship between $r$ and $f$ with BOP (Boundary Operating Point) and constraints.
**Numeric Results**

**FSI Case:**
**Full State Information:**
Controller is \( u(t) = Lx(t) \)

**PSI Case:**
**Partial State Information:**
One Velocity Sensor.
Controller is \( u(t) = L\hat{x}(t) \)
Outline

• Review of Covariance Analysis and Controller Design
• Covariance Based Sensor Selection
• Profit Based Operating Point Selection
• Profit Based Sensor Selection
Combined MBOP & Sensor Selection

EDOR for Different Sensor Networks

Constraints

MBOP

OSSOP
Combined MBOP & Sensor Selection
(Problem Formulation)

\[
\begin{align*}
\max \quad & d_s^T \tilde{s}_{ss} + d_m^T \tilde{m}_{ss} + d_c^T \beta_c \\
\text{s.t.} \quad & 0 = A\tilde{s}_{ss} + B\tilde{m}_{ss}; \quad \tilde{s}_{ss} = D_x \tilde{s}_{ss} + D_u \tilde{m}_{ss}, \quad \tilde{d}_{\min} \leq \tilde{s}_{ss} \leq \tilde{d}_{\max} \\
& \zeta_i < (\tilde{s}_{ss,i} - \tilde{d}_{\min,i})^2, \quad \zeta_i < (\tilde{s}_{ss,i} - \tilde{d}_{\max,i})^2, \quad i = 1 \ldots n_z \\
& 0 \leq \beta_{c,i} \leq 1, \quad 0 \leq \beta_{p,i} \leq 1 \quad i = 1 \ldots n_\beta \\
& \beta_{p,i} \leq h(\beta_{c,i}) \quad i = 1 \ldots n_\beta \quad \Sigma_v^{-1} = \text{diag}\left\{\beta_{p,i} / \overline{\sigma}_v^2\right\} \\
& \zeta_i = \phi_i [(D_x + D_u L)(\Sigma_x - \Sigma_e)(D_x + D_u L)^T \\
& \quad + D_x \Sigma_e D_x^T + D_w \Sigma_w D_w^T] \phi_i^T, \quad i = 1 \ldots n_z \\
& 0 = A\Sigma_x + \Sigma_x A^T + G\Sigma_w G^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^TB^T \\
& 0 = A\Sigma_e + \Sigma_e A^T - \Sigma_e C^T \Sigma_v^{-1} \Sigma_e + G\Sigma_w G^T
\end{align*}
\]
Combined MBOP & Sensor Selection
(Problem Formulation)

\[
\begin{align*}
\max_{\tilde{s}_{ss}, \tilde{m}_{ss}, \tilde{z}_{ss}, \\
\xi, \beta_c, \beta_p, \Sigma \geq 0, \Sigma_x \geq 0, \Sigma_e \geq 0}
& d_s^T \tilde{s}_{ss} + d_m^T \tilde{m}_{ss} + d_c^T \beta_c \\
\text{s.t.} \\
0 &= A \tilde{s}_{ss} + B \tilde{m}_{ss} \\
\tilde{z}_{ss} &= D_x \tilde{s}_{ss} + D_u \tilde{m}_{ss} \\
\tilde{d}_{\text{min}} &\leq \tilde{z}_{ss} \leq \tilde{d}_{\text{max}} \\
\zeta_i &< (\tilde{z}_{ss,i} - \tilde{d}_{\text{min},i})^2, \\
\zeta_i &< (\tilde{z}_{ss,i} - \tilde{d}_{\text{max},i})^2, \\
i &= 1 \cdots n_z \\
0 &\leq \beta_{c,i} \leq 1, \\
0 &\leq \beta_{p,i} \leq 1 \\
i &= 1 \cdots n_{\beta} \\
\beta_{p,i} &\leq h(\beta_{c,i}) \\
i &= 1 \cdots n_{\beta} \\
\Sigma_v^{-1} &= \text{diag}\left\{ \beta_{p,i} / \sigma^2_{v_i} \right\} \\
\zeta_i &= \phi_i [(D_x + D_u L)(\Sigma_x - \Sigma_e)(D_x + D_u L)^T \\
&+ D_x \Sigma_e D_x^T + D_w \Sigma_w D_w^T ]\phi_i^T, \\
i &= 1 \cdots n_z \\
0 &= A \Sigma_x + \Sigma_x A^T + G \Sigma_w G^T + BL(\Sigma_x - \Sigma_e) + (\Sigma_x - \Sigma_e)L^T B^T \\
0 &= A \Sigma_e + \Sigma_e A^T - \Sigma_e C^T \Sigma_v^{-1} C \Sigma_e + G \Sigma_w G^T
\end{align*}
\]
CSTR Example

Assume the current network ($SN_0$) consists of 6 sensors located at $C_{Ai}$, $C_A$, $T$, $V$, $F$, $P$ each having a precision of 2%.
CSTR Example

Upgrade Data:

- New 1% precision sensors can replace 2% sensors.
- Replacement cost is $1000/yr (includes purchase, installation, maintenance and replacement costs).
- 1% sensor available for other locations ($1000/yr )

Profit Function ($/yr):

\[ p(C_A, F, F_c, F_{vg}) = M_{an} \left[ \alpha_1 (C_{Ai} - C_A) F - \alpha_2 F_c - \alpha_3 F_{vg} \right] \]
## CSTR Example

<table>
<thead>
<tr>
<th>No</th>
<th>New Sensors</th>
<th>Profit ($/yr)</th>
<th>Sensor Costs ($/yr)</th>
<th>Profit - Sensor Costs ($/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_A$, $P$</td>
<td>36,030</td>
<td>2,000</td>
<td>34,030</td>
</tr>
<tr>
<td>2</td>
<td>$C_A$, $T_c$, $P$</td>
<td>36,600</td>
<td>3,000</td>
<td>33,600</td>
</tr>
<tr>
<td>3</td>
<td>$P$</td>
<td>33,470</td>
<td>1,000</td>
<td>32,470</td>
</tr>
<tr>
<td>4</td>
<td>$T$, $P$</td>
<td>34,390</td>
<td>2,000</td>
<td>32,390</td>
</tr>
<tr>
<td>5</td>
<td>$C_A$, $T$, $T_c$, $V$, $P$</td>
<td>37,060</td>
<td>5,000</td>
<td>32,060</td>
</tr>
<tr>
<td>6</td>
<td>$T$, $T_c$, $V$, $P$</td>
<td>35,120</td>
<td>4,000</td>
<td>31,120</td>
</tr>
<tr>
<td>7</td>
<td>none</td>
<td>28,970</td>
<td>0</td>
<td>28,970</td>
</tr>
</tbody>
</table>
CSTR Example

- Δ - Existing Network
- □ - Upgrade 5 Sensors
- ○ - Optimal Upgrade
CSTR Example

- Existing Network
- Upgrade 5 Sensors
- Optimal Upgrade
Conclusions

• Proposed New Profit Based Sensor Selection Scheme for Controlled Systems.

• Developed A Globally Optimal Search Algorithm.
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