On the Stability of Infinite Horizon Economic MPC

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Presentation Outline

- Motivation
- Example of Unstable EMPC
- Economic Linear Optimal Control
- Constrained ELOC
- Infinite Horizon EMPC
Inventory Creep

IGCC with Dispatch Capability

\[ v_{\text{coal}} \]

\[ v_{H_2} \]

\[ v_{H_2,S} \]

\[ v_{H_2,G} \]

Gasification Unit

Storage Tank

Power Block

Energy Value ($/MWhr)

\begin{align*}
\text{time (days)} & \\
\text{PJM Western Hub, Day-Ahead prices: June 6-10, 2001} & \\
\end{align*}
EMPC Applied to IGCC

- Energy Value ($/MWhr)
- Generated Power (MW)
- H₂ in Storage (tonnes)

Time (days): 6 to 10

Graphs showing variations over time for energy value, generated power, and hydrogen storage.
Buildings with Thermal Energy Storage

Heat from Environment → Building → Heat from Building → Heat to Chiller → Chiller → Power Consumption

Heat to TES

Thermal Energy Storage
Building EMPC and Inventory Creep

The graph shows the heat load to the chiller (in kW) and the energy in storage (in MWhr) over time (in hours). The labels for the y-axes are:

- **Heat Load to Chiller (kW)**
- **Energy in Storage (MWhr)**

The x-axis represents time (in hours) from 0 to 140. The graph includes two sets of data:

- **1 hr horizon**
- **12 hrs horizon**

The graph illustrates the dynamic response of the system over time, with distinct patterns for the two horizons.
Motivation

Does Inventory Creep Exist in Traditional Chemical Processes?
Presentation Outline

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- **Example of Unstable EMPC**
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Review of MPC

At time the current time, \( i \), solve:

\[
\min_{s_{k|i}, m_{k|i}} \left\{ \sum_{k=i}^{i+N-1} l(s_{k|i}, m_{k|i}, p_{k|i}) + l_f (s_{i+N|i}) \right\} \quad \text{s.t.}
\]

\[
\begin{align*}
  s_{k+1|i} &= f(s_{k|i}, m_{k|i}, p_{k|i}) \\
  q_{k|i} &= h(s_{k|i}, m_{k|i}, p_{k|i}) \\
  q_{\min} &\leq q_{k|i} \leq q_{\max} \quad k = i \ldots i + N - 1 \\
  s_{i|i} &= s_i
\end{align*}
\]

Then, the manipulated variable is set as: \( m_i = m_{i|i}^* \)
Traditional vs. Economic MPC

In traditional MPC the objective is regulation:

$$\min_{x_{ki}, u_{ki}} \left\{ \sum_{k=i}^{i+N-1} (x_{k|i}^T Q x_{k|i} + u_{k|i}^T R u_{k|i}) + x_{i+N|i}^T P x_{i+N|i} \right\}$$

In EMPC the objective is maximize average profit (integral of instantaneous profit):

$$\max_{s_{ki}, m_{ki}} \left\{ \sum_{k=i}^{i+N-1} g^{(IP)} (s_{k|i}, m_{k|i}, p_{k|i}) \right\}$$
Economic MPC Literature

- **Conceptual Development and Stability Issues**: Rawlings and Amrit (2009); Diehl, et al. (2011); Huang and Biegler (2011); Heidarinejad, et al. (2012)

- **Process Scheduling**: Karwana and Keblisb (2007); Baumrucker and Biegler (2010); Lima et al. (2011); Kostina et al. (2011)

- **Building HVAC Systems**: Braun (1992); Morris et al. (1994); Kintner-Meyer and Emery (1995); Henze et al. (2003); Braun (2007); Oldewurtel et al. (2010), Ma et al. (2012); Mendoza and Chmielewski (2012)

- **Power Scheduling**: Zavala et al. (2009); Xie and Ilić (2009), Hovgaard, et al. (2011), Omell and Chmielewski (2013)
Economic MPC Example

A non-isothermal CSTR:

\[
\frac{dC_A}{dt} = \frac{\nu}{V} (C_{A0} - C_A) - k_0 e^{\left(-\frac{E}{RT}\right)} C_A
\]

\[
\frac{dC_B}{dt} = -\frac{\nu}{V} C_B + k_0 e^{\left(-\frac{E}{RT}\right)} C_A
\]

\[
\frac{dT}{dt} = \frac{\nu}{V} (T_0 - T) + \left(\frac{\Delta H_r}{\rho C_p}\right) k_0 e^{\left(-\frac{E}{RT}\right)} C_A
\]

\[ T \leq 390 K \]

Disturbance: \( C_{A0} = C_{A0}^{ss} \pm 0.15 C_{A0}^{ss} \)

Manipulation: \( T_0 \leq 385 K \)


c_{A0}, T_0, \nu

\[
g^{(IP)}(C_B) = 10 \nu C_B
\]
Economic MPC Example

\[
\max_{s_{k|i}, T_{0,k|i}} \left\{ \sum_{k=i}^{i+N-1} 10^v C_{B,k|i} \right\} \\
\text{s.t.} \\
\begin{align*}
    s_{k+1|i} &= f(s_{k|i}, T_{0,k|i}, C_{A0,k|i}) \\
    s_{k|i} &= \begin{bmatrix} C_{A,k|i} & C_{B,k|i} & T_{k|i} \end{bmatrix}^T \\
    T_{k|i} &\leq 390K \\
    T_{0,k|i} &\leq 385K \\ 
    & k = i \ldots i + N - 1 \\
    S_{i|i} &= S_i \\
    N &= 8
\end{align*}
\]

*Model converted to discrete-time using Euler’s explicit method.*
Economic MPC Simulation

![Graphs showing temperature and concentration over time.](image)

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Economic MPC Simulation

![Graphs showing the concentration of species A and B over time.](image)

- **C_A (k mole/m^3)**
- **C_B (k mole/m^3)**

**Change to:**

- **C_A0 (k mole/m^3)**

**Legend:**

- **time (minutes)**
- **C_A (k mole/m^3)**
- **C_B (k mole/m^3)**

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Economic MPC Example

A non-isothermal CSTR:

\[ C_{A_0}, T_0, \nu \]

\[ C_A, C_B, T \]
Economic MPC Example

A non-isothermal CSTR with preheater:

\[ T_f = 303K \]

\[ C_{Ao}, T_0, \nu \]

\[ Q \]

\[ C_A, C_B, T \]

Manipulation: \( 303K \leq T_0 \leq 385K \)

Instantaneous Profit:

\[
g^{(IP)}(C_B,T_0) = \nu [10C_B - 0.005(T_0 - 303)] \approx \nu [5 - 0.4]
\]
Economic MPC Simulation

Graphs showing the evolution of concentration ($C_{A0}$) and temperature ($T$) over time in minutes.
Economic MPC Simulation

\[ g^{(IP)}(C_B, T_0) = \nu [10C_B - 0.005(T_0 - 303)] \]
Economic MPC Simulation

Change horizon from $N = 8$ to $N = 10$
Stability of EMPC

1) If $N=8$ and objective $\nu[10 C_B]$, then STABLE

2) If $N=8$ and objective $\nu[10 C_B - 0.005(T_0 - 303)]$, then UNSTABLE

3) If $N=10$ and objective $\nu[10 C_B - 0.005(T_0 - 303)]$, then STABLE

4) If $N=10$ and objective $\nu[10 C_B - 0.025(T_0 - 303)]$, then UNSTABLE

5) If $N=25$ and objective $\nu[10 C_B - 0.025(T_0 - 303)]$, then STABLE
Why the Instability?

\[ T_f = 303K \]

\[ Q \]

\[ \nu \left[ 10C_B - 0.025(T_0 - 303) \right] \]

**Answer: Myopic Behavior**

- EMPC tries to lower costs by lowering \( T_o \)
- In the short run \( T, C_B \) and Profit will remain high
- In the long run \( T, C_B \) and Profit will drop-off
- If \( N \) is small, then EMPC will not know about the drop-off
- Short term gains lead to long term losses
Summary of CSTR Example

Observations:
• Objective function influences stability
• Larger horizon improves stability
• Larger horizon increases computational burden

Conclusions:
• Consider infinite horizon formulation
• Exploit analytic simplicity of infinite horizon formulation
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- Motivation
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- **Economic Linear Optimal Control**
- Constrained ELOC
- Infinite Horizon EMPC
Economic Linear Optimal Control

\[ u_i = L_{ELOC} x_i \]
CSTR Example

\[ T_f = 303K \]

\[ C_{Ao}, T_o, \nu \]

\[ C_A, C_B, T \]

\[ C_B (\text{kmol/m}^3) \]

\[ C_A (\text{kmol/m}^3) \]
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**Linear Quadratic Regulator**

\[
\begin{aligned}
\min_{x_{k|i}, u_{k|i}} & \left\{ i+N-1 \sum_{k=i}^{i+N-1} (x_{k|i}^T Q x_{k|i} + u_{k|i}^T R u_{k|i}) + x_{i+N|i}^T P x_{i+N|i} \right\} \\
\text{s.t.} & \quad x_{k+1|i} = A x_{k|i} + B u_{k|i} \\
& \quad x_{i|i} = x_i \\
& \quad u_i = L_{LQR} x_i
\end{aligned}
\]

**Predictive Form of ELOC**

\[
\begin{aligned}
\min_{x_{k|i}, u_{k|i}} & \left\{ i+N-1 \sum_{k=i}^{i+N-1} (x_{k|i}^T Q_{ELOC} x_{k|i} + u_{k|i}^T R_{ELOC} u_{k|i}) + x_{i+N|i}^T P_{ELOC} x_{i+N|i} \right\} \\
\text{s.t.} & \quad x_{k+1|i} = A x_{k|i} + B u_{k|i} \\
& \quad x_{i|i} = x_i \\
& \quad u_i = L_{ELOC} x_i
\end{aligned}
\]

* see Chmielewski & Manthanwar (2004) for details
Constrained ELOC

\[
\begin{align*}
\min_{x_{k|i}, u_{k|i}} \left\{ \sum_{k=i}^{i+N-1} \left( x_{k|i}^T Q_{ELOC} x_{k|i} + u_{k|i}^T R_{ELOC} u_{k|i} \right) + x_{i+N|i}^T P_{ELOC} x_{i+N|i} \right\} \\
\text{s.t.} \\
\begin{align*}
x_{k+1|i} &= Ax_{k|i} + Bu_{k|i} \\
x_{i|i} &= x_i \\
z_{k|i} &= D_x x_{k|i} + D_u u_{k|i} \\
z_{\min} &\leq z_{k|i} \leq z_{\max}
\end{align*}
\end{align*}
\]
Constrained ELOC

\[ T(K) \]

\[ T_0(K) \]
Constrained ELOC

[Graph showing temperature (T) over time for EMPC and Constrained ELOC]
Constrained ELOC and Horizon Size

![Graph showing temperature T₀ (K) against time with EMPC and constrained ELOC curves for N=1 and N=10.]
Constrained ELOC and Nonlinear Plants

\[
\min_{x_{k|i}, u_{k|i}} \left\{ \sum_{k=i}^{i+N-1} (x_{k|i}^T Q_{ELOC} x_{k|i} + u_{k|i}^T R_{ELOC} u_{k|i}) + x_{i+N|i}^T P_{ELOC} x_{i+N|i} \right\} \quad s.t.
\]

\[
s_{k+1|i} = f(s_{k|i}, m_{k|i}) \quad s_{i|i} = s_i
\]

\[
q_{k|i} = h(s_{k|i}, m_{k|i})
\]

\[
q^{\text{min}} \leq q_{k|i} \leq q^{\text{max}}
\]
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EMPC with Infinite Horizon & ELOC Tail

\[
\text{min} \sum_{k=i}^{\infty} c^T z_{k|i} + \sum_{k=i+N}^{\infty} c^T z_{k|i} \quad \text{s.t.}
\]

\[
x_{k+1|i} = A x_{k|i} + B u_{k|i}
\]

\[
z_{k|i} = D_x x_{k|i} + D_u u_{k|i}
\]

\[
z^\text{min} \leq z_{k|i} \leq z^\text{max} \quad k = i \ldots \infty
\]

\[
x_{i|i} = x_i
\]

\[
u_{k|i} = L_{ELOC} x_{k|i} \quad k = i + N \ldots \infty
\]
Calculation of the Final Cost

\[ \sum_{k=i+N}^{\infty} C^T z_{k|i} \]
Calculation of the Final Cost

\[
\min_{k=i}^{i+N-1} \sum c^T z_{k|i}
\]

s.t. \( x_{k+1|i} = f(x_{k|i}, u_{k|i}) \)

\( z_{k|i} = h(x_{k|i}, u_{k|i}) \)

\( z_{min} \leq z_{k|i} \leq z_{max} \quad k = i \ldots i+N-1 \)

\( x_{i|i} = x_i \)
Calculation of the Final Cost

\[
\min \sum_{k=i}^{i+N-1} c^T z_{k|i} + (c_{ELOC})^T z_{i+N|i}
\]

s.t.

\[
x_{k+1|i} = f(x_{k|i}, u_{k|i})
\]

\[
z_{k|i} = h(x_{k|i}, u_{k|i})
\]

\[
z_{\min} \leq z_{k|i} \leq z_{\max} \quad k = i \ldots i+N-1
\]

\[
x_{i|i} = x_i
\]
Calculation of the Final Cost

\[
\min \sum_{k=i}^{i+N-1} c^T z_{k|i} + (c_{ELOC})^T z_{i+N|i}
\]

s.t. \( x_{k+1|i} = f(x_{k|i}, u_{k|i}) \)
\( z_{k|i} = h(x_{k|i}, u_{k|i}) \)
\( z_{\min} \leq z_{k|i} \leq z_{\max} \) \( k = i \ldots i + N - 1 \)
\( x_{i|i} = x_i \)
Calculation of the Final Cost

\[
\min \sum_{k=i}^{i+N-1} c^T z_{k|i} + (c_{ELOC})^T z_{i+N|i}
\]

s.t. \((x_{k+1|i} = f(x_{k|i}, u_{k|i}))\)

\(z_{k|i} = h(x_{k|i}, u_{k|i})\)

\(z_{\text{min}} \leq z_{k|i} \leq z_{\text{max}} \quad k = i \ldots i + N - 1\)

\(x_{i|i} = x_{i}\)
Conclusions

- **EMPC**
  - Stability may depend on objective function parameters
  - Larger horizons may provide stability

- **For CSTR example**
  - Constrained ELOC approximated EMPC
  - Infinite Horizon EMPC reproduced EMPC
  - Both provided stability, even with very small horizons
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