Constrained Infinite-time Optimal Control

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Outline

Background and Motivation

- Objectives of Constrained Control
- Finite-time Predictive Control

Constrained Infinite-time Optimal Control

- Advantages of Infinite-time Formulation
- Finite-time Solution Method

Constrained Disturbance Attenuation

- Stochastic Formulation
- Min Max Formulation
Naphtha Hydrotreating Process

From FCC

From Column

Recycle Ratio (MV2 + CV2)

MV4

Hydrogen Feed

Heat Exchanger

Vent Position (MV3)

Feed Temp Loop (MV2)

Fuel Feed

O₂ out (CV6)
CO out (CV8)

Reactor Temperature (CV4)

Hydrotreating Reactors

Clean Product
Modern Control System Architecture

Real Time Optimizer
(Steady State Calculations)

Predictive Control

Servo Loops

PID

PI

Process 1

Predictive Control

Servo Loops

PID

PI

Process 2

Predictive Control

Servo Loops

PID

PI

Process 3
Objectives of Real-time Optimization

Min Feed Rate

Max Feed Rate

Min Reactor Temperature

Max Reactor Temperature

Max CO Emissions

Expected Domain of Operation

Target Steady State

Operator Optimum

Economic Optimum

Steady State Target
Constrained Set Point Tracking

Min Feed Rate

Max Feed Rate

Min Reactor Temperature

Max Reactor Temperature

Max CO Emissions

Old Optimum

New Optimum

Old Target

New Target
Set Point Tracking

- Design a Feedback Controller so that the process follows the given Set Point Trajectory
PI Control System Design

Time Constant ($\tau$): Large
PI Gain ($K$): Small

Time Constant ($\tau$): Small
PI Gain ($K$): Large
Quadratic Optimal Control Design

Output Concern ($Q_{oc}$): Small
Move Suppression ($R_{ms}$): Large

Output Concern ($Q_{oc}$): Large
Move Suppression ($R_{ms}$): Small
Linear Quadratic Optimal Control

Problem Formulation

$$\min_{u_k, y_k, x_k} \left\{ \sum_{k=0}^{N} (y_k - y_{tgt})^T Q_{oc} (y_k - y_{tgt}) + \sum_{k=0}^{N-1} (u_k - u_{tgt})^T R_{ms} (u_k - u_{tgt}) \right\}$$

subject to

Process Model:

$$\begin{align*}
x_{k+1} &= Ax_k + Bu_k & 0 < k < N \\
y_k &= Cx_k & 0 \leq k \leq N
\end{align*}$$

Input Constraints:

$$u \leq u_k \leq \bar{u} & 0 \leq k < N$$

Output Constraints:

$$\underline{y} \leq y_k \leq \bar{y} & 0 < k \leq N$$
Constrained LQ Optimal Control

\[
\min_{u_k, x_k} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \left( \tilde{x}_k^T \tilde{Q}_{oc} \tilde{x}_k + \tilde{u}_k^T R_{ms} \tilde{u}_k \right) + \tilde{x}_N^T \tilde{Q}_{oc} \tilde{x}_N \right\}
\]

subject to

Process Model:
\[
\begin{align*}
\tilde{x}_{k+1} &= A \tilde{x}_k + B \tilde{u}_k \\
\tilde{x}_0 &= -x_{tgt}
\end{align*}
\]

Input Constraints:
\[
\underline{u} \leq \tilde{u}_k \leq \bar{u} \quad 0 \leq k < N
\]

Output Constraints:
\[
\underline{y} \leq C \tilde{x}_k \leq \bar{y} \quad 0 < k \leq N
\]

where: \( \tilde{Q}_{oc} = C^T Q_{oc} C \), \( x_{tgt} = A x_{tgt} + B u_{tgt} \), \( y_{tgt} = C x_{tgt} \), and
\[
\begin{align*}
\tilde{x}_k &= x_k - x_{tgt} \\
\bar{y} &= \bar{y} - y_{tgt} \\
\bar{u} &= \bar{u} - u_{tgt}
\end{align*}
\]
\[
\begin{align*}
\tilde{x}_k &= u_k - u_{tgt} \\
\bar{y} &= y - y_{tgt} \\
\bar{u} &= u - u_{tgt}
\end{align*}
\]
Trajectory Predictions vs. Actual

Present

Past ← Future

Predicted Output

Actual Output

Predicted Input

\[ y_k \]

\[ u_k \]

\[ \Delta t \]

\[ 2\Delta t \]

\[ 3\Delta t \]

\[ 4\Delta t \]

\[ 5\Delta t \]
Model Mismatch

\[ y_k^{(sp)} \]

\[ \hat{P} - \text{Process Model} \]

\[ P - \text{Actual Process} \]

\[ \Delta P = P - \hat{P} \]
Predictive Control

Past → Future

New Predicted Output

New Predicted Input

\( y_k \)

\( u_k \)

\( \Delta t \)

Present

Past → Future

New Predicted Output

New Predicted Input

\( \Delta t \)

2\( \Delta t \)

3\( \Delta t \)

4\( \Delta t \)

5\( \Delta t \)
Set Point Change in Hydrotreating Process

- **Feed Rate (MV1+CV1)**
- **Temperature (CV4)**
- **Recycle Ratio (MV2 + CV2)**
- **O₂ out (CV6)**
- **CO out (CV8)**
- **Vent Position (MV3)**
- **Hydrogen Feed (H₂ Sep)**
- **Clean Product**

**From FCC**
- **From Column**
- **Hydrogen Feed**
- **Recycle Ratio (MV2 + CV2)**
- **Feed Temp Loop (MV2)**
- **Fuel Feed**
- **Feed Rate (MV1+CV1)**
- **Heat Exchanger**
- **Furnace**
- **Hydrotreating Reactors**
- **O₂ out (CV6)**
- **CO out (CV8)**
- **Vent Position (MV3)**
- **Reactor Temperature (CV4)**
- **Clean Product**
# State Space Model of Hydrotreating Unit

\[
A = \begin{bmatrix}
    a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & a_{76} & a_{77} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{98} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1010} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1111} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1212} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1313}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
    b_{11} & 0 & 0 & 0 \\
    b_{21} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & b_{44} & 0 \\
    0 & 0 & b_{54} & 0 \\
    0 & 0 & 0 & b_{64} \\
    b_{81} & b_{82} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & b_{102} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & b_{123} & 0 \\
    0 & 0 & b_{133} & 0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
    c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & c_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & c_{33} & c_{34} & c_{35} & c_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & c_{49} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{510} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{611} & c_{612} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{713} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{812} & 0
\end{bmatrix}
\]
Set Point Change in Hydrotreating Process

New Target Point:

- Reactor Exit Temperature: Reduce by 4 degrees
- Reactor Inlet Temperature: Reduce by 5 degrees
- Reactant Feed Rate: Unchanged

Quadratic Program Statistics:

- Horizon Size: $N = 3$
- # of Optimization Variables: $16 \cdot N = 48$
- # of Equality Constraints: $13 \cdot N = 39$
- # of Inequality Constraints: $12 \cdot N = 36$
Constrained Infinite-time LQ Optimal Control

\[
\min_{u_k, x_k} \left\{ \sum_{k=0}^{\infty} x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k \right\}
\]

subject to

Process Model: \[ x_{k+1} = Ax_k + Bu_k \quad k > 0 \]

Input Constraints: \[ u \leq u_k \leq \bar{u} \quad k \geq 0 \]

Output Constraints: \[ y \leq C x_k \leq \bar{y} \quad k > 0 \]
LQ Optimal Control

- The Unconstrained LQ Problem, both Finite and Infinite-time versions (Kalman, 1960)

- Introduced Constrained Predictive Control to the ChE Community (Shell Group - Cutler, Prett, et. al., 1980)

- Guaranteed Closed-loop Stability with a Constrained Predictive Control Algorithm (Rawlings and Muske, 1993)

- The Constrained Infinite-time LQ Problem (Sznaier and Damborg, 1987; Chmielewski and Manousiouthakis, 1996; and Scokaert and Rawlings, 1998)
Main Result 1 - Constrained Stability

- If there exists any constrained stabilizing input trajectories,
  then the solution to Infinite-time Problem will be stabilizing

- If there does not exist any constrained stabilizing inputs,
  then the Infinite-time Problem will be infeasible
Unconstrained Infinite-time LQ Optimal Control

Let $P$ satisfy the Riccati Equation:

$$P = A^T \left( P - PB \left( R_{ms} + B^T P B \right)^{-1} B^T P \right) A + Q_{oc}$$

Then the Solution to the Unconstrained Problem is

$$x_0^T Px_0 = \min_{u_k, x_k} \left\{ \sum_{k=0}^{\infty} x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k \right\}$$

subject to $x_{k+1} = Ax_k + Bu_k$
Finite-time Solution to the Infinite-time Problem

\[
\min \left\{ \sum_{k=0}^{\infty} (x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k) \quad \text{s.t.} \quad \begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
u &\leq u_k \leq \bar{u} \\
y &\leq Cx_k \leq \bar{y}
\end{align*} \right. \\
= \min \left\{ \sum_{k=0}^{N-1} (x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k) \quad \text{s.t.} \quad \begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
u &\leq u_k \leq \bar{u} \\
y &\leq Cx_k \leq \bar{y}
\end{align*} \right. \\
+ \min \left\{ \sum_{k=N}^{\infty} (x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k) \quad \text{s.t.} \quad \begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
u &\leq u_k \leq \bar{u} \\
y &\leq Cx_k \leq \bar{y}
\end{align*} \right. \}
\]

\[
? = \min \left\{ \sum_{k=0}^{N-1} (x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k) + x_N^T P x_N \quad \text{s.t.} \quad \begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
u &\leq u_k \leq \bar{u} \\
y &\leq Cx_k \leq \bar{y}
\end{align*} \right. \\
= \min \left\{ \sum_{k=0}^{N-1} (x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k) + x_N^T P x_N \quad \text{s.t.} \quad \begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
u &\leq u_k \leq \bar{u} \\
y &\leq Cx_k \leq \bar{y}
\end{align*} \right. \}
\]
Finite-time Solution to Infinite-time Problem

\[
\sum_{k=0}^{N-1} (x_k^2 + u_k^2) + \sum_{k=N^*}^{\infty} (x_k^2 + u_k^2) + P x_{N^*}^2
\]
Main Result 2 - Existence

If there exists a constrained stabilizing input trajectory, then there always exists $N^*(x_0) < \infty$ such that

$$
\min_{u_k, x_k} \left\{ \sum_{k=0}^{N-1} \left( x_k^T Q_{oc} x_k + u_k^T R_{ms} u_k \right) + x_N^T P x_N \right\}
$$

subject to

$$
x_{k+1} = A x_k + B u_k \quad 0 \leq k < N$$
$$u \leq u_k \leq \bar{u} \quad 0 \leq k < N$$
$$y \leq C x_k \leq \bar{y} \quad 0 < k \leq N$$

gives the solution to the Infinite-time Problem for all $N \geq N^*(x_0)$
Constrained Infinite-time Nonlinear Optimal Control

\[
\min_{u(t), x(t)} \left\{ \int_0^\infty \left( x^T Q(x) x + u^T R(x) u \right) \, dt \right\}
\]

subject to

Process Model: \( \dot{x} = f(x) + g(x)u \), \( x(0) = \xi \)

Input Constraints: \( u \leq u(t) \leq \bar{u} \) \( t \geq 0 \)

Output Constraints: \( \underline{y} \leq Cx(t) \leq \bar{y} \) \( t > 0 \)

**Unconstrained Solution:** Find \( J(x) \) to satisfy the Hamilton-Jacobi-Bellman Equation

\[
0 = x^T Q(x) x + \left( \frac{\partial J}{\partial x} \right)^T f(x) - \frac{1}{4} \left( \frac{\partial J}{\partial x} \right)^T B(x) R^{-1}(x) B^T(x) \left( \frac{\partial J}{\partial x} \right)
\]
Constrained Infinite-time Nonlinear Optimal Control

\[
\min_{u(t), x(t)} \left\{ \int_0^{t_f} \left( x^T Q_{oc}(x)x + u^T R_{ms}(x)u \right) \ dt \ + \ J(x(t_f)) \right\}
\]

subject to

Process Model: \[ \dot{x} = f(x) + g(x)u , \ x(0) = \xi \]

Input Constraints: \[ u \leq u(t) \leq \bar{u} \quad 0 \leq t \leq t_f \]

Output Constraints: \[ y \leq Cx(t) \leq \bar{y} \quad 0 \leq t \leq t_f \]
Disturbance Rejection

\[ C \rightarrow P \]

\[ \text{Process Disturbances} \]

\[ \text{Actual Output} \]

\[ \text{Measurement Noise} \]

\[ \text{Measured Output} \]
Disturbance Types: Reactor Example

Possible Disturbance Models

Possible Measurement of $C_{A_{in}}$
Standard Disturbance Attenuation Formulations

- **Min Max LQ Optimal Control** - Assumes Worst Case Scenario

\[
\begin{array}{c}
\min_{\mu(\cdot)} \max_{w_k, v_k} \left\{ \frac{\sum_{k=0}^{\infty} x_k^T Q x_k + \mu^T (y_k) R \mu (y_k)}{\sum_{k=0}^{\infty} (w_k^T w_k + v_k^T v_k)} \right. \\
\text{s.t.} \quad x_{k+1} = A x_k + B \mu (y_k) + w_k \\
y_k = C x_k + v_k
\end{array}
\]

- **Stochastic LQ Optimal Control** - Assumes Most Likely Scenario

\[
\begin{array}{c}
\min_{\mu(\cdot)} \left\{ \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} (x_k^T Q x_k + \mu^T (y_k) R \mu (y_k)) \right. \\
\text{s.t.} \quad x_{k+1} = A x_k + B \mu (y_k) + w_k \\
y_k = C x_k + v_k \\
E [w_k w_k^T] = \Sigma_w, \quad E [v_k v_k^T] = \Sigma_v
\end{array}
\]

\[
= \min_{\mu(\cdot)} \left\{ \lim_{N \to \infty} E [x_k^T Q x_k + \mu^T (y_k) R \mu (y_k)] \right. \\
\text{s.t.} \quad x_{k+1} = A x_k + B \mu (y_k) + w \\
y_k = C x_k + v_k \\
E [w_k w_k^T] = \Sigma_w, \quad E [v_k v_k^T] = \Sigma_v
\end{array}
\]
Unconstrained Stochastic Optimal Control

Separation Principle:

A Cascaded Implementation of the Optimal Estimator followed by the Optimal Controller does not sacrifice Overall Optimality.

Certainty Equivalence:

Feedback Implementation of the Disturbance Free Optimal Controller provides the Solution to the Stochastic Optimal Control Problem.
Separation Principle

\[ y^{(sp)}_k \to + \to - \to \text{Stochastic Optimal Controller} \to u_k \to P \to y_k \to + \to + \to v_k \]

\[ y^{(sp)}_k \to + \to - \to \text{Optimal State Estimator} \to \hat{x}_k \to \text{FSI Stoc Optimal Controller} \to u_k \to P \to y_k \to + \to + \to v_k \]
Certainty Equivalence

If \( J_\Sigma \) satisfies the Hamilton-Jacobi-Bellman Equation ( \( \mathcal{E} [w w^T] = \Sigma \) )

\[
J_\Sigma(x) = J'_\Sigma(x) - \inf_x \{ J'_\Sigma(x) \}
\]

\[
J'_\Sigma(x) = \min_{\underline{u} \leq u \leq \bar{u}} \left\{ x^T Q'_oc x + u^T R_{ms} u + \mathcal{E} [ J_\Sigma(Ax + Bu + w) ] \right\}
\]

Then the full information stochastic optimal feedback policy is

\[
\mu_\Sigma(x) = \arg \min_{\underline{u} \leq u \leq \bar{u}} \left\{ u^T R_{ms} u + \mathcal{E} [ J_\Sigma(Ax + Bu + w) ] \right\}
\]

In the Unconstrained Case, Certainty Equivalence holds ( i.e., \( \mu_\Sigma(x) = \mu_0(x) \) ).

In the Constrained Case, \( \mu_\Sigma(x) \approx \mu_0(x) \) only if the disturbances are small.
Constrained Stochastic Optimal Control
Main Results

- Separation Principle Continues to hold in the Constrained Case

- Certainty Equivalence Does Not hold in the Constrained Case

- If the Open-Loop Process (i.e., the $A$ matrix) is Stable, then the Constrained Stochastic Optimal Policy Exists

- If the Open-Loop Process (i.e., the $A$ matrix) is Unstable, then the Constrained Stochastic Optimal Policy Does Not Exist

- Output Constraints can be Incorporated through a Soft Constraints
Disturbance Rejection: CSTR Example

Operating Conditions:

- Volumetric Flow Rate ($\nu_o$): 0.4 (m³/min)  \(0.325 \leq \nu_o \leq 0.475\)
- Reaction Rate Constant ($k_1$): 0.06 (m³/kmole min)
- Residence Time ($\tau$): 5 (min)
- Concentration of A In ($C_{A_{in}}$): 10 (kmole/m³)
Constrained Worst Case Problem Formulation

\[
\begin{align*}
\min_{\underline{u}} & \leq \mu(\cdot) \leq \overline{u} & \max_{\underline{w}} & \leq \omega(\cdot) \leq \overline{w} & \left\{ \sum_{k=0}^{\infty} H(x_k) \text{ s.t. } x_{k+1} = Ax_k + B\mu(x_k) + \omega(x_k) \right\} \\
H(x) &= \begin{cases} 
x^TQx + \mu^T(x)\mu(x) - \gamma^2\omega^T(x)\omega(x) & \text{if } x \in X_{\text{max}} \\
\infty & \text{if } x \notin X_{\text{max}}
\end{cases}
\end{align*}
\]

where

- \( X_{\text{max}} \) is the constrained stabilizable set for the disturbance free problem
- The Nonlinear Operators \( \mu(\cdot) \) and \( \omega(\cdot) \) are the Optimization Variables
- \( \gamma > 0 \) is a fixed constant
Constrained Min Max Optimal Control

If $J$ satisfies the Isaacs’ Equation

$$J(x) = \min_{\underline{u} \leq u \leq \overline{u}} \max_{\underline{w} \leq w \leq \overline{w}} \left\{ x^TQx + u^Tu - \gamma^2w^Tw + J(Ax + Bu + w) \right\}$$

s.t. $Ax + Bu + w \in X_{\text{max}}$

Then the Min Max Optimal Feedback policy is

$$\begin{bmatrix} \mu(x) \\ \omega(x) \end{bmatrix} = \arg \min_{\underline{u} \leq u \leq \overline{u}} \max_{\underline{w} \leq w \leq \overline{w}} \left\{ u^Tu - \gamma^2w^Tw + J(Ax + Bu + w) \right\}$$

s.t. $Ax + Bu + w \in X_{\text{max}}$
A Finite-time Max Min Problem

\[
\begin{bmatrix}
\mu(x_0) \\
\omega(x_0)
\end{bmatrix} = \arg \max_{w_k} \min_{u_k, x_k} \left\{ \begin{array}{l}
\sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T u_k - \gamma^2 w_k^T w_k) + x_N^T P x_N \\
\text{s.t. } x_{k+1} = A x_k + B u_k + w_k \\
y \leq C x_k \leq \bar{y} \\
u \leq u_k \leq \bar{u} \\
w \leq w_k \leq \bar{w}
\end{array} \right\}
\]

where

\[
P = Q + A^T P A^{-1} A \quad \text{and} \quad \Lambda = I + (B B^T - \gamma^{-2} G G^T) P
\]
Min Max Example

Consider the One State System:

\[ x_{k+1} = 0.9x_k + u_k + w_k \]

With Constraints:

\[ |u_k| \leq 0.3 \text{ , and } |w_k| \leq 0.05 \]

And Design Parameters:

\[ Q = 1 \text{ , and } \gamma^2 = \sqrt{3} \]
Conclusions

- The CITLQOC Controller has many Advantages over the Finite-time Predictive Controller (i.e., Tuning and Stability)

- The CITLQOC can be solved using a Finite-Dimensional Optimization Problem

- The Constrained Stochastic Optimal Controller has Performance Advantages, with a Substantial Increase in Computational Cost

- The Constrained Min Max Controller also has Performance Advantages, with a only Moderate Increase in Computational Cost
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