Z angew Math Phys 47 (1996)
(ZAMP)

0044-2275/96/030432-24 $ 1.50 + 0.20

The effect of nonlocal interactions on the dynamics of the Ginzburg–Landau equation

By Jinqiao Duan, California Institute of Technology, Applied Mechanics 104-44, Pasadena, CA 91125, USA. E-mail: duan@galcit.caltech.edu, Hung Van Ly, Department of Mathematics, University of California, Irvine, CA 92717, USA. E-mail: hly@math.uci.edu and Edriss S. Titi, Department of Mathematics, and Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92717, USA. E-mail: etiti@math.uci.edu

Abstract

Nonlocal amplitude equations of the complex Ginzburg–Landau type arise in a few physical contexts, such as in ferromagnetic systems. In this paper, we study the effect of the nonlocal term on the global dynamics by considering a model nonlocal complex amplitude equation. First, we discuss the global existence, uniqueness and regularity of solutions to this equation. Then we prove the existence of the global attractor, and of a finite dimensional inertial manifold. We provide upper and lower bounds to their dimensions, and compare them with those of the cubic complex Ginzburg–Landau equation. It is observed that the nonlocal term plays a stabilizing or destabilizing role depending on the sing of the real part of its coefficient. Moreover, the nonlocal term affects not only the diameter of the attractor but also its dimension.

1. Introduction

Amplitude equations are approximate reductions of complicated physical systems on extended domains near bifurcation points (see Vega (1993), Newell et al. (1993) and references therein). Amplitude equations usually have "universal" forms and therefore, one amplitude equation may capture important features of various physical systems. The derivation of such amplitude equations is usually based on a formal asymptotic expansion with respect to a small parameter. A rigorous justification for such derivation is presented in Takáč et al. (1995) (see also references therein).

The nonlinear Schrödinger equation and the complex Ginzburg–Landau equation are examples of amplitude equations, which are in local forms. The nonlinear Schrödinger equation comes from conservative physical systems while the Ginzburg–Landau equation comes from dissipative ones. In some situations nonlocal integral terms may appear in the amplitude equations. For example, if the original physical system has some kind of symmetries, integral terms may appear in the derived amplitude equation (Matkowsky and Volpert (1992)). Some derivation methods may also introduce integral terms (Mielke (1992)). When the original physical systems contain nonlocal terms, the reduced or derived amplitude equations may contain similar nonlocal terms as in Elmer (1988). Elmer (1988) derived the following amplitude equation for a class of physical systems including a ferromagnetic system:

$$u_t = a_0 u + (a_1 + ib_1)u_{xx} + (a_2 + ib_2)|u|^2 u + (a_3 + ib_3)u \frac{1}{l} \int_0^l |u|^2 \, dx, \quad (1.1)$$

subject to the periodic boundary condition

$$u(x + l, t) = u(x, t), \quad (1.2)$$

and supplemented with initial condition

$$u(x, 0) = u_0(x), \quad (1.3)$$

for some fixed $l > 0$, and for every $x \in \mathbb{R}, t > 0$. The function $u(x, t)$ is complex and the coefficients a_j's, b_j's are real. When $a_3 = b_3 = 0$ the equation (1.1) reduces to the cubic complex Ginzburg–Landau equation,

$$u_t = a_0 u + (a_1 + ib_1)u_{xx} + (a_2 + ib_2)|u|^2 u. \quad (1.4)$$

Other nonlocal integro-differential amplitude equations also appear in the modeling of interfacial phenomena (Papageorgiou et al. (1990)), frontal phenomena (Wilder et al. (1994) and Metzener et al. (1994)), and bifurcation phenomena (Or-Guil et al. (1994)).

A global attractor for an evolutionary partial differential equation is the maximal bounded invariant set, which attracts all solution orbits in the phase space or functional space. The global attractor needs not to be a smooth subset of the phase space and it may be very complicated or even fractal. An inertial manifold for an evolutionary equation is, on the other hand, a finite dimensional Lipschitz manifold, which is positively invariant under the solution flow, and which exponentially attracts all bounded sets in the phase space. See Foias–Sell–Temam (1988), Foias, Sell and Titi (1989), Temam (1988), Hale (1988), and Constantin et al. (1989) for more details. As a result it necessarily contains the global attractor. The flow restricted to an inertial manifold is equivalent to that of a finite dimensional system of ordinary differential equations called an inertial form. An inertial form can be regarded as a global "amplitude" equation.

This paper is organized as follows. In sections 2, 3, we discuss the problem of global existence, uniqueness and regularity of solutions to the nonlocal equation (1.1), and consider the dissipativity, i.e., estimate the size of absorbing sets. In section 4, we prove the existence of a global attractor, provide upper and lower bound estimates for the dimension of the global attractor, and compare them with those of the cubic complex Ginzburg–Landau equation. In section 5, we show the existence of a finite-dimensional inertial manifold for this equation, and estimate, from above, its dimension. Notice that the lower bound for the dimension of the global attractor forms a lower bound to the dimension of the inertial manifold as well. We observe that these bounds on the dimensions of the attractor and of the inertial manifold change dramatically as a function of the coefficients of the nonlocal term. In section 6 we discuss the concepts of determining nodes and other degrees of freedom. In particular, we show, following the work of Kukavica (1992), that there are two determining nodes for this nonlocal equation. In section 7 we consider the special case of (1.1), when $a_2 = b_2 = 0$. In this case we are able to construct explicitly the Inertial Manifold of lowest dimension. Finally we end the paper with a discussion in section 8.

2. Global existence and regularity of solutions

We rescale the spatial variable x to obtain that the periodic interval length $l = 1$. In the following, the integrals are with respect to $x \in [0, 1]$ unless it is specified otherwise, and $\|\cdot\|$ denotes the L^2-norm. For physical reasons we are interested in bounded solutions, i.e., solutions in L^∞. Since in one spatial dimension L^∞ is embedded in the Sobolev space H^1, it is sufficient, for simplicity, to pose the problem in H^1 (see remarks about other kinds of regularities later). Let $A = -\partial^2/\partial x^2$ with periodic boundary conditions and let $N(u)$ denote the remaining terms of the right hand side of (1.1). Then we can write (1.1) as

$$\frac{du}{dt} + (a_1 + ib_1)Au = N(u) \tag{2.1}$$

with initial data

$$u(x, 0) = u_0(x). \tag{2.2}$$

Since the Sobolev space H^1 in one spatial dimension is an algebra, it is easy to show that $N\colon H^1_{\text{per}}(0, 1) \to H^1_{\text{per}}(0, 1)$ is a locally Lipschitz map. In fact for any $u, v \in H^1_{\text{per}}(0, 1)$

$$\begin{aligned}\|N(u) - N(v)\|_{H^1} &= \|a_0(u - v) + (a_2 + ib_2)(|u|^2 u - |v|^2 v) \\ &\quad + (a_3 + ib_3)(u\|u\|^2 - v\|v\|^2)\|_{H^1} \\ &\le C_1 \|u - v\|_{H^1},\end{aligned}$$

where c_1 is a constant depends on $(a_0, a_2, b_2, a_3, b_3, \|u\|_{H^1}, \|v\|_{H^1})$. Thus for $a_1 > 0$ the local existence in H^1 can be obtained via the semigroup theory (Pazy (1983) Theorem 6.3.1, or Henry (1981) Theorem 3.3.3). That is, if $u_0 \in H^1_{\mathrm{per}}(0, 1)$, then there exists $T^* = T^*(\|u_0\|_{H^1_{\mathrm{per}}(0,1)}) > 0$ such that (2.1) and (2.2) has unique local solution $u \in C([0, T^*); H^1_{\mathrm{per}}(0, 1)) \cap C^1((0, T^*); H^1_{\mathrm{per}}(0, 1))$.

Moreover, one can also easily show that $N\colon H^k_{\mathrm{per}}(0, 1) \to H^k_{\mathrm{per}}(0, 1)$ is a locally Lipschitz map for any positive integer k. Therefore, within the short time of existence $0 < t < T^*$, we can also show that the solution indeed belongs to $H^k_{\mathrm{per}}(0, 1)$ for any positive integer k (see Henry (1981), page 73). In particular, for any $0 < t < T^*$ the solution belongs to $C^\infty(0, 1)$. For further discussion concerning the regularity of the solution see the Remark at the end of this section.

First observe that when $a_2 + a_3 > 0$ there are solutions which blow up in finite time. Indeed, let us seek a solution to the equation (1.1) of the form $u(x, t) = U(t)$. Hence, $U(t)$ satisfies

$$\dot{U} = a_0 U + [(a_2 + a_3) + i(b_2 + b_3)]|U|^2 U.$$

Let us multiply the above equation by $\bar{U}$ and take the real part to get

$$\frac{1}{2}\frac{d}{dt}|U|^2 = a_0|U|^2 + (a_2 + a_3)|U|^4. \tag{2.3}$$

It is clear that certain solutions to this equation will blow up in finite time, provided that $a_2 + a_3 > 0$. Therefore a necessary condition for the global existence is $a_2 + a_3 \leq 0$. Using argument similar to the ones below one can show the global existence of solution for the case $a_2 + a_3 = 0$. However, in this case one can see from (2.3) that all spatially homogeneous solutions except zero will grow exponentially for $a_0 > 0$ and hence the system is not dissipative. Therefore we will concentrate on the case for $a_2 + a_3 < 0$ from now on unless otherwise specified. By assuming further that $a_2 < 0$, we are going to show that the supremum of the H^1-norm of the solution remains bounded uniformly over any interval of time for which the solution exists. From this we infer that the solution exists for all $t > 0$. Suppose that the solution exists on the maximal interval of existence $[0, T_{\max})$. We need to show that $T_{\max} = \infty$. To this end it is enough to show that the H^1-norm of the solution is bounded uniformly on the interval $[0, T_{\max})$. This will be done by straight forward energy estimates. Let us first establish the uniform bound for the L^2-norm of u. Take the real part of the L^2 inner product of the equation (1.1) with $\bar{u}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = a_0\|u\|^2 - a_1\|u_x\|^2 + a_2\int_0^1 |u(x, t)|^4\, dx + a_3\|u\|^4. \tag{2.4}$$

Since we assumed $a_2 < 0$ then $a_2 \|u\|_{L^4}^4 \leq a_2 \|u\|_{L^2}^4$ and (2.4) becomes

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 \leq a_0\|u\|^2 - a_1\|u_x\|^2 + (a_2 + a_3)\|u\|^4. \tag{2.5}$$

Note that since we assumed $a_1 > 0$ and $a_2 + a_3 < 0$, hereafter we assume that $a_0 > 0$ otherwise the dynamics is trivial. Using Young's inequality

$$a_0\|u\|^2 \leq |a_2 + a_3|\,\|u\|^4 + \frac{a_0^2}{4|a_2 + a_3|},$$

and so we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + a_1\|u_x\|^2 \leq \frac{a_0^2}{4|a_2 + a_3|}. \tag{2.6}$$

Let $K_0 = a_0^2/(4|a_2 + a_3|)$, if we ignore the term $a_1\|u_x\|^2$ on the left hand side, equation (2.6) gives us

$$\|u(t)\|^2 \leq \|u_0\|^2 + 2K_0 T_{\max}, \tag{2.7}$$

for every $t \in [0, T_{\max})$. Moreover by integrating equation (2.6) with respect to τ over $(s, t) \subset [0, T_{\max})$, we obtain

$$\frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|u(s)\|^2 + a_1\int_s^t \|u_x(\tau)\|^2\,d\tau \leq K_0(t - s).$$

Consequently by using (2.7) we have,

$$a_1\int_0^{T_{\max}} \|u_x(\tau)\|^2\,d\tau \leq \frac{1}{2}\|u_0\|^2 + 2K_0 T_{\max} := K_1, \tag{2.8}$$

where K_1 depends on $\|u_0\|$ and $T_{\max}$. We now establish the uniform bound for $\|u_x\|$. Based on our previous remarks about the regularity of the solution we know that the solution is at least in H^2, for $t \in (0, T_{\max})$. We multiply equation (1.1) with $\bar{u}_{xx}$, integrate with respect to x over the interval $[0, 1]$ and then take the real part to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2 &= a_0\|u_x\|^2 - a_1\|u_{xx}\|^2 + 2a_2\int_0^1 |u|^2|u_x|^2\,dx \\ &\quad + \mathrm{Re}\left[(a_2 + ib_2)\int_0^1 u^2\bar{u}_x^2\,dx\right] + a_3\|u\|^2\|u_x\|^2.\end{aligned} \tag{2.9}$$

By using a series of Agmon's and Young's inequalities,

$$\int_0^1 |u|^2|u_x|^2 \leq \|u\|_{L^\infty}^2\|u_x\|^2 \leq \frac{3}{2}\|u\|^2\|u_x\|^2 + \frac{1}{2}\|u_x\|^4. \tag{2.10}$$

Because of the uniform bound (2.7), equation (2.9) can be estimated as follows

$$\frac{d}{dt}\|u_x\|^2 + 2a_1\|u_{xx}\|^2 \leq C_2(1+\|u_x\|^2)^2 \tag{2.11}$$

where $C_2 = C_2(a_0, a_2, b_2, a_3, \|u_0\|, T_{\max})$. Let $y := 1 + \|u_x\|^2$ so equation (2.11) becomes

$$\frac{dy}{dt} \leq C_2 y^2. \tag{2.12}$$

In addition based on (2.8) we have

$$\int_0^{T_{\max}} y(\tau)\,d\tau \leq T_{\max} + K_1 := K_2. \tag{2.13}$$

By a generalized version of the Gronwall's inequality (see, for instance, Foias–Sell–Temam (1988) and Temam (1988)) equations (2.12) and (2.13) imply that

$$y(t) \leq K_3, \tag{2.14}$$

for some positive constant K_3 which depends on $\|u_0\|$ and $T_{\max}$. Hence (2.7) and (2.14) imply that $\|u(t)\|_{H^1_{\text{per}}(0,1)}$ is uniformly bounded for $t \in [0, T_{\max})$. Therefore we can extend the solution to any $t > 0$ (Pazy (1983)), and we obtain the global existence of mild solutions in H^1. Thus we have proved the following theorem:

Theorem 1. Assume that $a_1 > 0$, $a_2 \leq 0$ and $a_2 + a_3 \leq 0$. Then for every initial data $u_0 \in H^1_{\text{per}}(0, 1)$ there exists a unique solution u to the problem (1.1)–(1.2)–(1.3), which is defined globally for all $t > 0$. Moreover, this solution is analytic in space and time for all $t > 0$. If in addition we assume $a_2 + a_3 < 0$, then the solution is bounded uniformly in time.

Remark. Alternatively, one can use the Galerkin procedure to prove existence and regularity of solutions. Actually, under the same condition, we can show that weak solutions exist for $u_0 \in L^2_{\text{per}}(0, 1)$ via the Galerkin method. In particular, following the work of Ferrari and Titi (1994), which is based on Foias and Temam (1989), one can show that the solution belongs to a Gevrey class of regularity and as a result it is real analytic with respect to the spatial variable x (see also Doelman and Titi (1993), Duan, Holmes and Titi (1993)). Moreover, one can also implement the approach of Foias and Temam (1979) (see also Constantin and Foias (1988) and Foias and Temam (1989)) to show that the solution is analytic with respect to the temporal variable t as well. For further results concerning the existence uniqueness and regularity of a class of "local" complex Ginzburg–Landau equations see Levermore and Oliver (1995).

3. Radii of the absorbing balls

From now on we will assume that $a_2 + a_3 < 0$ and $a_2 + ib_2 \neq 0$. We will discuss in full details the case when $a_2 = b_2 = 0$ in section 7. In this section we will follow the work of Doering et al. (1988) to get estimates for the radii of the absorbing balls. Moreover we will show that our estimates agree with that of the cubic Ginzburg–Landau equation in Doering et al. (1988) when the coefficient of the nonlocal term is zero. In the previous section §2, when we established global existence, our estimate for the L^2-norm was rough. We will now give a sharp estimate for the size of the absorbing ball in L^2 (see the expression for the Stokes solution below (4.26)). By dropping the term $-a_1 \|u_x\|^2$ on the right hand side of equation (2.5), we have

$$\frac{d}{dt} \|u(t)\|^2 \leq 2\{a_0 + (a_2 + a_3)\|u(t)\|^2\}\|u(t)\|^2. \tag{3.1}$$

This implies that

$$\limsup_{t \to \infty} \|u(t)\| \leq \varrho_0 := \sqrt{\frac{a_0}{|a_2 + a_3|}}, \tag{3.2}$$

which means that the equation has an absorbing ball in L^2 of radius ϱ_0. Let us denote by

$$\delta = \max\{0, (\sqrt{a_2^2 + b_2^2} + 2a_2)\}, \tag{3.3}$$

then equation (2.9) implies

$$\frac{1}{2}\frac{d}{dt} \|u_x\|^2 \leq a_0 \|u_x\|^2 - a_1 \|u_{xx}\|^2 + a_3 \|u\|^2 \|u_x\|^2 + \delta \|uu_x\|^2. \tag{3.4}$$

If we assume that $|b_2| \leq \sqrt{3}|a_2|$, i.e. $\delta = 0$, then equation (3.4) gives

$$\frac{1}{2}\frac{d}{dt} \|u_x\|^2 \leq a_0 \|u_x\|^2 - a_1 \|u_{xx}\|^2 + a_3 \|u\|^2 \|u_x\|^2.$$

Observe that

$$\|u_x\|^2 \leq \|u\| \, \|u_{xx}\|, \tag{3.5}$$

Thus from the above we get

$$\frac{1}{2}\frac{d}{dt} \|u_x\|^2 = \left(a_0 - a_1 \frac{\|u_x\|^2}{\|u\|^2} + a_3 \|u\|^2\right)\|u_x\|^2. \tag{3.6}$$

It follows from equation (3.2) and equation (3.6) that for $|b_2| \le \sqrt{3}|a_2|$ the absorbing ball in terms of $\|u_x\|$ has radius ϱ_1, where

$$\varrho_1 = a_0\sqrt{\left(\frac{1}{a_2|a_2+a_3|}\right)\left(1+\frac{a_3}{|a_2+a_3|}\right)}. \tag{3.7}$$

If $|b_2| > \sqrt{3}|a_2|$, i.e. $\delta > 0$, then equation (3.4) gives

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 = a_0\|u_x\|^2 - a_1\|u_{xx}\|^2 + \delta\|u\|^2_{L^\infty}\|u_x\|^2 + a_3\|u\|^2\|u_x\|^2. \tag{3.8}$$

Putting (3.5) in (3.8) we have

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 = \left(a_0 - a_1\frac{\|u_x\|^2}{\|u\|^2} + \delta\|u\|^2_{L^\infty} + a_3\|u\|^2\right)\|u_x\|^2. \tag{3.9}$$

It is clear that from simple calculus inequalities we have the following version of Agmon's inequality:

$$\|u\|^2_{L^\infty} \le \|u\|^2 + 2\|u\|\,\|u_x\|. \tag{3.10}$$

Thus it follows from (3.9) and 3.10) that

$$\frac{1}{2}\frac{d}{dt}\|u_x\|^2 = \left(a_0 - a_1\frac{\|u_x\|^2}{\|u\|^2} + \delta\|u\|^2 + 2\delta\|u\|\,\|u_x\| + a_3\|u\|^2\right)\|u_x\|^2. \tag{3.11}$$

Again after some delicate algebraic manipulation, we obtain an absorbing ball with radius ϱ_1' in terms of $\|u_x\|$

$$\varrho_1' = \frac{\delta a_0^{3/2}}{a_1|a_2+a_3|^{3/2}}\left[1+\sqrt{1+\frac{a_1|a_2+a_3|(|a_2+a_3|+\delta+a_3)}{\delta^2 a_0}}\right], \tag{3.12}$$

provided that $|b_2| > \sqrt{3}|a_2|$. Applying (3.2), (3.7), and (3.12) in (3.10), we obtain an absorbing ball in the L^∞ of radius ϱ_∞, where if $|b_2| \le \sqrt{3}|a_2|$

$$\varrho_\infty^2 = \frac{a_0}{|a_2+a_3|} + \frac{2a_0^{3/2}}{|a_2+a_3|}\sqrt{\frac{1}{a_1}\left(1+\frac{a_3}{|a_2+a_3|}\right)} \tag{3.13}$$

and

$$\varrho_\infty^2 = \frac{a_0}{|a_2+a_3|} + \frac{2\delta a_0^2}{a_1|a_2+a_3|^2}\left[1+\sqrt{1+\frac{a_1|a_2+a_3|(|a_2+a_3|+\delta+a_3)}{\delta^2 a_0}}\right], \tag{3.14}$$

otherwise. Doering et al. (1988) established the sizes of the absorbing balls for the cubic Ginzburg–Landau equation (1.4), i.e. the equation (1.1) without the nonlocal integral term. They obtained the following radii:

$$\varrho_0 \sim a_0^{1/2}$$

$$\varrho_1 \sim \begin{cases} a_0 & \text{if } |b_2| \le \sqrt{3}|a_2| \\ \delta a_0^{3/2}\left[1 + \sqrt{1 + \dfrac{1+\delta}{\delta^2 a_0}}\right] & \text{if } |b_2| > \sqrt{3}|a_2| \end{cases}$$

and

$$\varrho_\infty^2 \sim \begin{cases} a_0 + 2a_0^{3/2} & \text{if } |b_2| \le \sqrt{3}|a_2| \\ a_0 + 2\delta a_0^2\left[1 + \sqrt{1 + \dfrac{1+\delta}{\delta^2 a_0}}\right] & \text{if } |b_2\| > \sqrt{3}|a_2|, \end{cases}$$

where δ is given in (3.3). Evidently the sizes of the absorbing balls for the nonlocal amplitude equation are different from those of the cubic Ginzburg–Landau equation. However if we let $a_1 = 1$, $a_2 = -1$ and $a_3 = 0$ in (3.2), (3.7), (3.11), (3.13) and (3.14), then we have all radii exactly coincide, with those of Doering et al. (1988).

The above estimates on the L^∞ norm were obtained by using interpolation inequalities. Let us remark, however, that direct and probably sharper estimates might be obtained by implementing the techniques developed by Collet (1994).

4. Global attractor and dimension estimates

Here again we consider the case when $a_2 + a_3 < 0$ and $a_2 + ib_2 \neq 0$. Let us denote by $u(x, t) = S(t)u_0(x)$ the solution operator. Based on the Remark at the end of section 2 we conclude from (3.2), (3.7) and (3.12) that $S(t)$ is a bounded operator from $L^2_{\text{per}}(0, 1)$ into $H^1_{\text{per}}(0, 1)$, for every $t > 0$. Since $H^1_{\text{per}}(0, 1)$ is compactly imbedded in $L^2_{\text{per}}(0, 1)$, then $S(t)$ is a compact mapping from $L^2_{\text{per}}(0, 1)$ into $L^2_{\text{per}}(0, 1)$ for every $t > 0$. Thus the ball $B(0, \varrho_0) = \{w \in L^2(0, 1) \mid \|w\| < \varrho_0\}$ is an absorbing set in $L^2_{\text{per}}(0, 1)$ and the ω-limit set of $B(0, \varrho_0)$ under $S(t)$, which we denote $\mathscr{A}$,

$$\mathscr{A} = \bigcap_{s>0} \overline{\left(\bigcup_{t \ge s} S(t)B(0, \varrho_0) \right)}, \tag{4.1}$$

where the closure is taken in the L^2 topology, is the global attractor (see Hale (1988), Temam (1988) and references therein). The global attractor $\mathscr{A}$ is necessarily a nonempty compact subset of $L^2_{\text{per}}(0, 1)$.

We are going to follow the works of Constantin and Foias (1988), Constantin et al. (1985), Ghidaglia and Héron (1988), Doering et al. (1987), Temam (1988), Babin and Vishik (1992) (see also references therein) to obtain the upper and lower bounds of the dimension of $\mathscr{A}$. We use the trace formula of Constantin–Foias–Temam to find the dimension of the attractor of the complex Ginzburg–Landau equation with the extra nonlocal term. We note that the condition under which the trace formula of Constantin–Foias–Temam is valid is satisfied here (see Temam (1988)). We now determine the upper bound of the Hausdorff and fractal dimension of the global attractor $\mathscr{A}$. In the later part of this section, we will also establish a lower bound for the dimension of the global attractor.

By putting the nonlocal equation (1.1) in the form $u_t = F(u)$ and then linearize it around any fixed solution $u(x, t)$ in the global attractor we get $U_t = F'(u)U$; that is U satisfies the evolution equation:

$$U_t = a_0 U + (a_1 + ib_1)U_{xx} + (a_2 + ib_2)[|u|^2 U + 2u \operatorname{Re}(\bar{u}U)] \\ + (a_3 + ib_3)\left[2u \int_0^1 \operatorname{Re}(u\bar{U})\, dx + U \int_0^1 |u|^2\, dx\right] \tag{4.2}$$

with $U(0) = \xi \in L^2_{\text{per}}(0, 1)$ and $U(x, t)$ satisfies the same periodic boundary conditions as $u(x, t)$. Let $\xi = \xi_1, \ldots, \xi_m$ be m linearly independent functions of $L^2_{\text{per}}(0, 1)$. We solve (4.2) with initial conditions $U(x, 0) = \xi_i$, $i = 1, \ldots, m$, and denote the corresponding solutions by $U = U_1, \ldots, U_m$. We set $P_m(t)$ to be the orthogonal projection of $L^2_{\text{per}}(0, 1)$ onto the space spanned by $\{U_1(t), \ldots, U_m(t)\}$. At any given time τ, let $\{\phi_j(\tau)\}_{j=1}^m$ be a subset of H^1_{per} which is an orthonormal basis, with respect to the $L^2_{\text{per}}(0, 1)$ inner product, i.e. $(\phi_i(\tau), \phi_j(\tau)) = \delta_{ij}$, of the linear space $P_m(\tau)L^2_{\text{per}}$, i.e.

$$\operatorname{span}\{\phi_1(\tau), \ldots, \phi_m(\tau)\} = \operatorname{span}\{U_1(\tau), \ldots, U_m(\tau)\}.$$

Note that

$$Re\{\operatorname{Trac}(F'(u(t))P_m(t))\} = \sum_{j=1}^m Re(F'(u(t))\phi_j(t), \phi_j(t)), \tag{4.3}$$

and

$$Re(F'(u(t))\phi_j(t), \phi_j(t)) \\ = a_0 \|\phi_j(t)\|^2 + a_1 \left\| \frac{\partial \phi_j(t)}{\partial x} \right\|^2 \\ + Re\left\{(a_2 + ib_2) \int_0^1 (|u|^2|\phi_j(t)|^2 + 2u\bar{\phi}_j(t)\, Re(\bar{u}\phi_j(t)))\, dx\right\} \\ + Re\left\{(a_3 + ib_3)\left[2\left(Re \int_0^1 u\phi_j(t)\, dx\right) \int_0^1 u\bar{\phi}_j(t)\, dx + \|u\|^2 \|\phi_j(t)\|^2\right]\right\}. \tag{4.4}$$

But

$$|u|^2|\phi_j|^2 + 2u\bar{\phi}_j \operatorname{Re}(\bar{u}\phi_j) = 2|u|^2|\phi_j|^2 + u^2\bar{\phi}_j^2,$$

thus argue similarly to (3.4) we obtain

$$\operatorname{Re}\left\{(a_2 + ib_2) \int_0^1 (|u|^2|\phi_j(t)|^2 + 2u\bar{\phi}_j(t) \operatorname{Re}(\bar{u}\phi_j(t)))\, dx\right\}$$
$$\leq \delta \int_0^1 |u|^2|\phi_j|^2\, dx, \tag{4.5}$$

where δ is given by (3.3). Also

$$\operatorname{Re}\left\{(a_3 + ib_3)\left[2\left(\operatorname{Re}\int_0^1 u\phi_j(t)\, dx\right)\int_0^1 u\bar{\phi}_j(t)\, dx + \|u\|^2\|\phi_j(t)\|^2\right]\right\}$$
$$\leq 3\sqrt{a_3^2 + b_3^2}\,\|u\|^2\|\phi_j\|^2 \tag{4.6}$$

Inserting equations (4.5) and (4.6) in (4.4) and using the fact that $\{\phi_j(t)\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2_{\text{per}}(0, 1)$ at any given time t we obtain:

$$\operatorname{Re}(F'(u(t))\phi_j(t), \phi_j(t)) \leq a_0 - a_1\left\|\frac{\partial\phi_j}{\partial x}\right\|^2 + \delta\int_0^1 |u|^2|\phi_j|^2\, dx$$
$$+ 3\sqrt{a_3^2 + b_3^2}\|u\|^2.$$

Let us denote by $f = \Sigma_{j=1}^m |\phi_j|^2$. It follows that

$$\sum_{j=1}^m \operatorname{Re}(F'(u(t))\phi_j, \phi_j) \leq ma_0 - a_1\sum_{j=1}^m\left\|\frac{\partial\phi_j}{\partial x}\right\|^2 + \delta\int_0^1 |u(x,t)|^2 f(x,t)\, dx$$
$$+ 3m\sqrt{a_3^2 + b_3^2}\|u\|^2. \tag{4.7}$$

By Sobolev–Lieb–Thirring inequality (see Temam 1988, p. 466) there exist two absolute constants c_1 and c_2, which are independent of m and of the functions $\{\phi_j\}_{j=1}^m$, such that

$$\|f\|^3_{L^3(0,1)} \leq c_1\int_0^1 f(x)\, dx + c_2\sum_{j=1}^m\left\|\frac{\partial\phi_j}{\partial x}\right\|^2.$$

Using the fact that $\|\phi_j\| = 1$, it follows that

$$\int_0^1 f(x)\, dx = \sum_{j=1}^m \int_0^1 |\phi_j|^2\, dx = \sum_{j=1}^m \|\phi_j\|^2 = m,$$

and so we have

$$\|f\|^3_{L^3(0,1)} \leq c_1 m + c_2\sum_{j=1}^m\left\|\frac{\partial\phi_j}{\partial x}\right\|^2. \tag{4.8}$$

Substitute (4.8) in (4.7) to obtain

$$\sum_{j=1}^{m} \operatorname{Re}(F'(u(t))\phi_j, \phi_j) \le ma_0 + \frac{a_1 c_1 m}{c_2} - \frac{a_1}{c_2}\|f\|^3_{L^3(0,1)} + \delta \int_0^1 |u|^2 f\,dx + 3m\sqrt{a_3^2+b_3^2}\|u\|^2. \tag{4.9}$$

By using Hölder inequality, with $p = 3/2$ and $q = 3$, we get

$$\delta \int_0^1 |u(x,t)|^2 f(x,t)\,dx \le \delta \|u\|^2_{L^3(0,1)} \|f\|_{L^3(0,1)}.$$

We then apply Young's inequality to the right hand side to get

$$\delta \int_0^1 |u|^2 f\,dx \le \frac{a_1}{2c_2}\|f\|^3_{L^3(0,1)} + \sqrt{\frac{8\delta^3 c_2}{27a_1}}\|u\|^3_{L^3(0,1)}. \tag{4.10}$$

Putting (4.10) in (4.9) we have

$$\sum_{j=1}^{m} \operatorname{Re}(F'(u(t))\phi_j, \phi_j) \le \left(a_0 + \frac{a_1 c_1}{c_2} + 3\sqrt{a_3^2+b_3^2}\|u\|^2\right)m - \frac{a_1}{2c_2}\|f\|^3_{L^3(0,1)} + \sqrt{\frac{8\delta^3 c_2}{27a_1}}\|u\|^3_{L^3(0,1)}. \tag{4.11}$$

Note that since

$$m = \sum_{j=1}^{m} \|\phi_j\|^2 = \int_0^1 f\,dx \le \|f\|_{L^3(0,1)},$$

so that (4.11) becomes

$$\sum_{j=1}^{m} \operatorname{Re}(F'(u(s))\phi_j, \phi_j) \le \left(a_0 + \frac{a_1 c_1}{c_2} + 3\sqrt{a_3^2+b_3^2}\|u\|^2\right)m - \frac{1a_1 m^3}{2c_2} + \sqrt{\frac{8\delta^3 c_2}{27a_1}}\|u\|^3_{L^3(0,1)}. \tag{4.12}$$

Let us observe that on the attractor

$$a_0 + \frac{a_1 c_1}{c_2} + 3\sqrt{a_3^2+b_3^2}\|u\|^2 \le a_0 + \frac{a_1 c_1}{c_2} + 3\sqrt{a_3^2+b_3^2}\varrho_0^2 := K. \tag{4.13}$$

But from Young's inequality we have

$$Km \le \frac{a_1 m^3}{4c_2} + \sqrt{\frac{16c_2}{27a_1}}K^{3/2}.$$

Therefore by inserting the above inequality in (4.12), we get

$$\sum_{j=1}^{m} \operatorname{Re}(F'(u(t))\phi_j, \phi_j) \leq -\frac{a_1}{4c_2} m^3 + \sqrt{\frac{16c_2}{27a_1}} K^{3/2} + \sqrt{\frac{8\delta^3 c_2}{27a_1}} \|u\|^3_{L^3(0,1)}. \quad (4.14)$$

Because the nonlocal equation possesses the absorbing balls in various norms, for some s large enough equation (4.14) can be estimated as

$$\sum_{j=1}^{m} \operatorname{Re}(F'(u(t))\phi_j, \phi_j) \leq \sqrt{\frac{16c_2}{27a_1}} \left(a_0 + \frac{a_1 c_1}{c_2} + 3\sqrt{a_3^2 + b_3^2 \varrho_0^2} \right)^{3/2} + \sqrt{\frac{8\delta^3 c_2}{27a_1}} \varrho_\infty^3 - \frac{a_1}{4c_2} m^3. \quad (4.15)$$

where ϱ_0 and ϱ_∞ are defined in (3.2) and (3.13)–(3.14) respectively. Let

$$q_m = \limsup_{t \to \infty} \sup_{\substack{\xi_j \in L^2_{\text{per}}(0,1) \\ \|\xi_j\| \leq 1 \\ j=1,2,\ldots,m}} \frac{1}{t} \int_0^t \operatorname{Re}\{\operatorname{Trac}(F'(u(s))P_m(s))\}\, ds, \quad (4.16)$$

it follows that

$$q_m \leq -\kappa_1 m^3 + \kappa_2 \quad (4.17)$$

where

$$\kappa_1 = \frac{a_1}{4c_2} \quad (4.18)$$

and

$$\kappa_2 = \sqrt{\frac{16c_2}{27a_1}} \left(a_0 + \frac{a_1 c_1}{c_2} + 3\sqrt{a_3^2 + b_3^2 \varrho_0^2} \right)^{3/2} + \sqrt{\frac{8\delta^3 c_2}{27a_1}} \varrho_\infty^3. \quad (4.19)$$

Following the work of Costantin, Foias and Temam (1985) (see also Constantin and Foias (1988), Temam (1988) and references therein), the global attractor $\mathscr{A}$ has Hausdorff dimension $d_H(\mathscr{A})$ less than or equal to m and fractal dimension $d_F(\mathscr{A})$ less than or equal to $2m$ where m is defined as

$$m - 1 \leq \left(\frac{2\kappa_2}{\kappa_1} \right)^{1/3} \leq m. \quad (4.20)$$

Therefore, using (4.18) and (4.19) we find that

$$d_H(\mathscr{A}) \leq d_F(\mathscr{A}) = 4\sqrt{\frac{2c_2}{3a_1}} (K^{3/2} + \delta^{3/2} \varrho_\infty^3)^{1/3}. \quad (4.21)$$

where K, δ, ϱ_∞ are defined in (4.13), (3.3) and (3.13)–(3.14) respectively. As we can see, the upper bound (4.21) for the dimension of the global attractor of the nonlocal equation is very complicated. However we can see the upper bound changes with respect to the coefficients of the nonlocal term a_3 and

b_3. Moreover when the nonlocal term is absent, we can compare with the upper bound of the global attractor of the usual cubic Ginzburg–Landau equation. This is interpreted as follows:

1. In the case $|b_2| \le \sqrt{3}|a_2|$ (i.e. $\delta = 0$),

$$d_H(\mathscr{A}) \le c_1'\left(1 + \frac{a_0}{a_1} + \frac{a_0}{a_1}\frac{\sqrt{a_3^2 + b_3^2}}{|a_2 + a_3|}\right)^{1/2}. \tag{4.22}$$

For the usual cubic Ginzburg–Landau equation, Ghidaglia and Héron (1987) found

$$d_H(\mathscr{A}) \le c_1'\left(1 + \frac{a_0}{a_1}\right)^{1/2}. \tag{4.23}$$

Therefore by comparing with the usual cubic Ginzburg–Landau equation, we find that the difference between the square of the upper bounds of the dimensions of the global attractor $\mathscr{A}$ for the nonlocal equation and the cubic Ginzburg–Landau is equal to

$$\frac{c_1' a_0 \sqrt{a_3^2 + b_3^2}}{a_1 |a_2 + a_3|}. \tag{4.24}$$

2. For the case $|b_2| > \sqrt{3}|a_2|$ (i.e. $\delta > 0$), we can see from (4.20) that along with the increase of (4.24) as in case 1 above, the upper bound of the dimension of the global attractor $\mathscr{A}$ for the nonlocal equation is increased further due to the change of ϱ_∞. That is

$$d_H(\mathscr{A}) \le c_1''\left\{\left(1 + \frac{a_0}{a_1} + \frac{a_0\sqrt{a_3^2 + b_3^2}}{a_1|a_2 + a_3|}\right)^{1/2} + \left(\frac{a_0\delta}{a_1|a_2 + a_3|}\left(1 + \frac{2\delta}{|a_2 + a_3|}\right.\right.\right.$$
$$\left.\left.\left.\times\left[1 + \left(1 + \frac{a_1|a_2 + a_3|(|a_2 + a_3| + \delta + a_3)}{\delta^2 a_0}\right)^{1/2}\right]\right)\right)^{1/2}\right\}. \tag{4.25}$$

We now establish the lower bound of the dimension of the global attractor. Following Ghidaglia and Héron (1987), we linearize the equation around the Stokes solution

$$u_s(t) = \sqrt{\frac{a_0}{|a_2 + a_3|}}\, e^{ia_0((b_2 + b_3)/a_2 + a_3)t}, \tag{4.26}$$

and check the number of unstable eigenvalues for the corresponding linearized differential operator. This number is the dimension of the invariant unstable manifold $E_+(u_s(t))$. According to Theorem VII 3.2 of Temam (1988) (see also Babin and Vishik (1992) and references therein) $E_+(u_s(t))$ is contained in the global attractor $\mathscr{A}$, and hence the dimension of $E_+(u_s(t))$ gives a lower bound of the dimension of $\mathscr{A}$.

We make the following change of variables

$$u = u_s(1 + v),$$

and write the equation (1.1) in terms of v. We then linearize this equation at $v = 0$ in the direction of w to obtain:

$$w_t + Lw = 0,$$

where the linear operator L is given by:

$$Lw = (a_1 + ib_1)\left[Aw + \frac{a + ib}{2}(w + \bar{w})\right] + \frac{a_0(a_3 + ib_3)}{|a_2 + a_3|}\int_0^1 (w + \bar{w})\, dx,$$

with the constants

$$a = \frac{2a_0}{|a_2 + a_3|}\frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2},$$

$$b = \frac{2a_0}{|a_2 + a_3|}\frac{a_1 b_2 - b_1 a_2}{a_1^2 + b_1^2}.$$

We need to estimate the number of unstable eigenfunctions of the linear operator L, i.e., those which correspond to eigenvalues with positive real part. Let us denote by $w = R + iI$, and by $w_0 = R_0 + iI_0 = \int_0^1 w(x)\, dx$. The $Lw = (a_1 + ib_1)Tw$, where

$$Tw = Aw + (a + ib)R + \frac{a_0(a_3 + ib_3)}{|a_2 + a_3|(a_1 + ib_1)} R_0.$$

This should be understood as a real operator defined on the space of vector valued functions over the scalar field of real numbers. Namely, as a map

$$T(R, I) = \Bigg(AR + aR + \mathrm{Re}\left\{\frac{a_0(a_3 + ib_3)}{|a_2 + a_3|(a_1 + ib_1)}\right\}R_0,$$
$$AI + bR + \mathrm{Im}\left\{\frac{a_0(a_3 + ib_3)}{|a_2 + a_3|(a_1 + ib_1)}\right\}R_0\Bigg).$$

Notice that $\mathrm{spec}(L) = (a_1 + ib_1)\mathrm{spec}(T)$. Therefore it is enough to find the spectrum of T. It is easy to see that the space of constant vector valued functions is invariant subspace under T, and that T has only two constant linearly independent eigenfunctions corresponding to the eigenvalues $\{\mu = 0,\ \mu = a + \mathrm{Re}\{a_0(a_3 + ib_3)/|a_2 + a_3|(a_1 + ib_1)\}\}$. The rest of spectrum of T can be found, as in Ghidaglia and Héron (1987), to be $\bigcup_{j=0}^{\infty}\{j^2, j^2 + a\}$. As a result, we have:

$$\mathrm{spec}(T) = \bigcup_{j=1}^{\infty}\{j^2, (j^2 + a)\} \cup \left\{a + \mathrm{Re}\left\{\frac{a_0(a_3 + ib_3)}{|a_2 + a_3|(a_1 + ib_1)}\right\}\right\}.$$

We need to find the number of eigenvalues of L, including their multiplicities, which lie in the left half complex plane. When $a < 0$, i.e., when

$$a_1 a_2 + b_1 b_2 < 0,$$

there are unstable eigenvalues, and as in Ghidaglia and Héron (1987) or Temam (1988, page 406) we can estimate the lower bound of the dimension of the global attractor

$$\dim_H(\mathscr{A}) \geq \dim E_+(u_s) \geq -1 + 2\sqrt{\frac{2a_0|a_1 a_2 + b_1 b_2|}{|a_2 + a_3|(a_1^2 + b_1^2)}} - \min\left\{0, \operatorname{sgn}\left(a_1 a + \operatorname{Re}\left\{\frac{a_1 a_0(a_3 + ib_3)}{|a_2 + a_3|(a_1 + ib_1)}\right\}\right)\right\}. \tag{4.27}$$

This is a lower bound of the dimension of the global attractor $\mathscr{A}$. Note that we have assumed that $a_2 + a_3 < 0$ and $a_2 < 0$. Under these conditions, when compared with the corresponding cubic Ginzburg–Landau equation, the lower bound of the dimension of $\mathscr{A}$ is smaller when $a_3 < 0$, and larger when $a_3 > 0$. This means that the nonlocal term has global stabilizing and destabilizing effect on the global dynamics depending on the sign of the real part of its coefficient. Thus we have proved the following theorem.

Theorem 2. Assume that $a_1 > 0$, $a_2 \leq 0$ and $a_2 + a_3 < 0$. Then there exists a finite dimensional global attractor in $L^2_{\text{per}}(0, 1)$ for the dynamical system (1.1)–(1.2)–(1.3). The upper and lower bounds for the dimension of the global attractor are given in (4.21) and (4.27), respectively.

5. Inertial manifold and its dimension estimate

In this section we will first show the existence of inertial manifold of the nonlocal amplitude equation (1.1), then we will give an explicit estimate to its dimension. We will also compare this dimension estimate with that of the usual cubic Ginzburg–Landau equation; in order to find evidence of the effect of the nonlocal term.

The existence and dimension estimate for an inertial manifold of the usual cubic Ginzburg–Landau equation is studied in Constantin et al. (1989) and Doering et al. (1988). We are going to prove the existence of an inertial manifold for the nonlocal amplitude equation (1.1) using the spectral barrier method of Constantin et al. (1989). This can also be done following the arguments in Sell and You (1992) or Debussche and Temam (1993). Here again we consider the case when $a_2 + a_3 < 0$ and $a_2 + ib_2 \neq 0$. Let $A_0 = -a_1(\partial^2/\partial x^2)$ and $B_0 = -ib_1(\partial^2/\partial x^2)$ with periodic boundary conditions. Since $a_1 > 0$, A_0 is a self-adjoint non-negative definite operator on

$\mathscr{D}(A_0) = H^2_{\text{per}}$. B_0 commutes with A_0 and is antisymmetric with respect to the inner product $(\cdot\,,\cdot)$ and the semi-inner product $(A_0\,\cdot\,,\cdot)$. That is

$$(B_0 u,\ v) = -(u,\ B_0 v)$$

$$(B_0 A_0 u,\ v) = -(u,\ A_0 B_0 v).$$

Let $N_0(u) = a_0 u + (a_2 + ib_2)|u|^2 u + (a_3 + ib_3)u \int_0^1 |u|^2\,dx$, then (1.1) can be written as

$$u_t + (A_0 + B_0)u = N_0(u). \tag{5.1}$$

Let $\Theta(r)$ be a smooth non-increasing function on $[0, \infty)$ such that $\Theta(r) = 1$ for $r \le 1$ and $\Theta(r) = 0$ for $r \ge 3$, and $|\Theta'(r)| \le 1$ for all $r \in [0, \infty)$. Also we let

$$\Psi(u) = \Theta\left(\frac{\|u\|_{L^\infty}}{\varrho_\infty}\right)\Theta\left(\frac{\|u\|}{\varrho_0}\right).$$

Since we are interested in the long time behavior of the solutions then we will concentrate on the behavior at the absorbing ball. Therefore, we will truncate $N_0(u)$ outside the absorbing balls, and consider the prepared equation of (1.1)

$$u_t + (A_0 + B_0)u = \Psi(u)N_0(u) \tag{5.2}$$

instead of equation (5.1) itself. Let

$$R(u) = \Psi(u)N_0(u),$$

then one can easily verify the following estimates

$$\begin{aligned}\|R(u) - R(v)\| &\le (7a_0 + 81\sqrt{(a_2^2 + b_2^2)}\varrho_\infty^2 + 13\sqrt{(a_3^2 + b_3^2)}\varrho_0^2)\|w\| + (3a_0 \\ &\quad + 27\sqrt{(a_2^2 + b_2^2)}\varrho_\infty^2 + 3\sqrt{(a_3^2 + b_3^2)}\varrho_0^2)\|w\|^{1/2}\|A^{1/2}w\|^{1/2} \\ &\le \mu_1\|w\| + \mu_1\|w\|^{1/2}\|A^{1/2}w\|^{1/2},\end{aligned} \tag{5.3}$$

where $w = u - v$ and $\mu_1 = (7a_0 + 81\sqrt{(a_2^2 + b_2^2)}\varrho_\infty^2 + 13\sqrt{(a_3^2 + b_3^2)}\varrho_0^2)$.

We call a number $\lambda \in (0, \infty)$ a *spectral barrier* for the equation (1.1) if for every two distinct solutions $u, v \in \mathscr{D}(A)$ satisfying

$$(A(u - v), u - v) = \lambda\|u - v\|^2,$$

It follows that

$$\|(A - \lambda)(u - v)\|^2 + (R(u) - R(v), (A - \lambda)(u - v)) > 0.$$

We remark here that a spectral barrier cannot be an eigenvalue of A. Furthermore, it follows from (5.3) that in order for λ to be a spectral barrier, it is sufficient that

$$\|(A-\lambda)(w)\| \geq (\mu_1 + \mu_1 \lambda^{1/4}) \|w\|. \tag{5.4}$$

So if we let λ be the midpoint $\frac{1}{2}(\lambda_{N+1}+\lambda_N)$ of two consecutive eigenvalues of A, then

$$\|(A-\lambda)(w)\| \geq \frac{1}{2}(\lambda_{N+1}-\lambda_N)\|w\|,$$

and hence equation (5.4) is satisfied if

$$\frac{1}{2}(\lambda_{N+1}-\lambda_N) \geq (\mu_1 + \mu_1 \lambda^{1/4}).$$

In other words

$$N + \frac{1}{2} \geq \frac{\mu_1}{4\pi^2}(1+\lambda^{1/4}). \tag{5.5}$$

On the other hand, the existence of inertial manifolds also requires that

$$\lambda > 2\max\{\lambda_{0,1}, \lambda_{0,3/4}\}, \tag{5.6}$$

for $\lambda_{0,1}$, $\lambda_{0,3/4}$ satisfying

$$(R(u)-R(v),(u-v)) \geq -\lambda_{0,1}\|u-v\|^2 \\ - \lambda_{0,3/4}(\|u-v\|^{3/4}\|A^{1/2}(u-v)\|^{(1-(3/4))})^2.$$

In our case, it follows from equation (5.3) that

$$\lambda_{0,1} = \mu_1 \quad \text{and} \quad \lambda_{0,3/4} = \mu_1^{4/3}. \tag{5.7}$$

So equation (5.6) is equivalent to

$$\lambda > 2\max\{\mu_1, \mu_1^{4/3}\}. \tag{5.8}$$

Therefore we have from (5.5) and (5.8) that there exists an inertial manifold $\mathcal{M}$ whose dimension $\dim(\mathcal{M})$ satisfies:

$$\dim(\mathcal{M}) + \frac{1}{2} > \frac{\mu_1}{4\pi^2}(1 + (2\max\{\mu_1, \mu_1^{4/3}\})^{1/4}), \tag{5.9}$$

where for $|b_2| \leq \sqrt{3}|a_2|$,

$$\mu_1 = \left(7 + 81\frac{(a_2^2+b_2^2)^{1/2}}{|a_2+a_3|} + 13\frac{(a_3^2+b_3^2)^{1/2}}{|a_2+a_3|}\right)a_0 \\ + 162\frac{(a_2^2+b_2^2)^{1/2}}{|a_2+a_3|}\sqrt{\frac{1}{a_1}\left(1+\frac{a_3}{|a_2+a_3|}\right)}a_0^{3/2}, \tag{5.10}$$

and for $|b_2| > \sqrt{3}|a_2|$,

$$\mu_1 = \left(7 + 81\frac{(a_2^2+b_2^2)^{1/2}}{|a_2+a_3|} + 13\frac{(a_3^2+b_3^2)^{1/2}}{|a_2+a_3|}\right)a_0 + 162\delta\frac{(a_2^2+b_2^2)^{1/2}}{|a_2+a_3|^2}$$
$$\times\left[1+\sqrt{1+\frac{a_1|a_2+a_3|(|a_2+a_3|+\delta+a_3)}{\delta^2 a_0}}\right]a_0^2. \tag{5.11}$$

In conclusion we have the following theorem:

Theorem 3. Let $a_1 > 0$, $a_2 \leq 0$ and $a_2 + a_3 < 0$. Then there exists an inertial manifold $\mathcal{M}$ for the dynamical system (1.1)–(1.2)–(1.3) whose dimension is specified in (5.9)–(5.10)–(5.11).

We can see the relation of the upper bound of the dimension of the inertial manifold for equation (1.1) with that for the corresponding cubic Ginzburg–Landau equation (1.4), by considering various values of a_3. Primarily, if $a_3 < 0$, then it follows from (5.10) and (5.11) that the dimension of the inertial manifold for the nonlocal amplitude equation will be smaller than that of the cubic Ginzburg–Landau equation. This is actually what we would expect because the nonlocal term now behaves like a stabilizing term. On the other hand if $0 < a_3 < |a_2|$, then the upper bound of the dimension of the inertial manifold will be larger. This is, due to the now-defective (or now-destabilizing) non-local term, and the long time dynamics of the equation (1.1) is described by a higher dimensional ordinary differential system. Moreover as $a_3 \nearrow |a_2|$, the upper bound of the dimension of the inertial manifold will get enormously large. This is obviously true because when $a_2 = -a_3$ the equation is no longer dissipative. Indeed, as it is clearly seen from equation (2.3) in this case we are no longer guaranteed the existence of an absorbing ball. Our last emphasis is that these estimates on the dimensions depend on the sizes of the absorbing balls so when the sizes of the absorbing balls increase/decrease, not only the diameters of the inertial manifolds or the global attractor get larger/smaller but also their dimensions get larger/smaller as well.

6. Determining nodes and other degrees of freedom

A finite set of points $\mathscr{E} = \{x_1, x_2, \ldots, x_N\} \subset [0, l]$ is called determining if for any two solutions u, v of (1.1) for which $\max_{1\leq i\leq N}|u(x_i, t) - v(x_i, t)| \to 0$, we have $\|u - v\| \to 0$. This notion of degrees of freedom was first introduced by Foias and Temam (1984) in the context of the Navier–Stokes equations. For estimates to the number of determining nodes and other kinds of degrees of freedom for the Navier–Stokes equations see

Jones and Titi (1992, 1993), Cockburn, Jones and Titi (1995) and references therein. Kukavica (1992) has shown that for the cubic complex Ginzburg–Landau equation (1.4) there are two determining nodes. The proof is based on the fact that we have one spatial variable and that the solutions of equation (1.4) belong to a Gevrey class of regularity, and hence are real analytic. Let us observe again that following the work of Ferrari and Titi (1994), which is based on the work of Foias and Temam (1989), we can show that the solutions of the nonlocal complex Ginzburg–Landau equation (1.1) belong to a Gevrey class of regularity, and are real analytic. Then, following the work of Kukavica (1992) we can show that equation (1.1) has two determining nodes. In Section 5 it was shown that the inertial manifolds are parameterized by the Fourier modes. In Foias and Titi (1991) it was shown that the inertial manifolds for many equations, such as Kuramoto–Sivashinsky equation and the complex Ginzburg–Landau equation (1.1), can be parameterized by the nodal values of the solutions or by the local volume elements. Moreover, it is shown that the number of nodes or volume elements needed for such parameterization is proportional to the dimension of the inertial manifold. This kind of parameterization was recently generalized by Cockburn, Jones and Titi (1995) to include finite elements and other kinds of interpolant polynomials. We observe that these results are valid for the nonlocal complex Ginzburg–Landau equation (1.1). We omit the details because they are word by word as in Foias and Titi (1991) and Cockburn, Jones and Titi (1995).

7. The case $a_2 = b_2 = 0$

In this section we consider equation (1.1) with $a_2 = b_2 = 0$, i.e.

$$u_t = a_0 u + (a_1 + ib_1)u_{xx} + (a_3 + ib_3)u \frac{1}{l} \int_0^l |u|^2 \, dx, \tag{7.1}$$

subject to the same periodic boundary condition. Again we rescale the spatial variable to obtain that the periodic interval length $l = 1$. For the global existence it is now required that $a_3 \leq 0$; otherwise as shown in section 2 for sufficiently large initial data the solution will blow up in finite time. In addition, in the case of $a_3 < 0$ one can follow the argument in Section 3 to obtain estimates on the sizes of the absorbing balls in L_2, L_2 of u_x, and L^∞ norms for equation (7.1). However, this is a simple case in which we will be able to construct explicitly the Inertial Manifold of dimension equals to that of the global attractor. The details here are in the spirit of the work of Bloch and Titi (1990). Notice that $\{e^{2\pi kxi}\}_{k \in Z}$ is a complete set in L^2_{per} of eigenfunctions of the operator A. Let $H_m = \text{span}\{e^{2\pi kxi}: -m \leq k \leq m\}$ and $P_m: L^2_{\text{per}}(0, 1) \to H_m$ be the L_2 orthogonal projection onto H_m. If we write the solution of (7.1) in the form

$$u = p + q,$$

where $p = P_m u$ and $q = (I - P_m)u$, then equation (7.1) becomes

$$p_t = a_0 p + (a_1 + ib_1)p_{xx} + (a_3 + ib_3)\left(\int_0^1 |u|^2\, dx\right)p \tag{7.2}$$

$$q_t = a_0 q + (a_1 + ib_1)q_{xx} + (a_3 + ib_3)\left(\int_0^1 |u|^2\, dx\right)q. \tag{7.3}$$

The real part of the L^2 inner product of the equation (7.3) with $\bar{q}$ gives

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 = a_0\|q\|^2 - a_1\|q_x\|^2 - |a_3|\,\|u\|^2\|q\|^2.$$

Since $\|q_x\|^2 \geq (2\pi m)^2\|q\|^2$, we have

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 \leq (a_0 - a_1(2\pi m)^2)\|q\|^2. \tag{7.4}$$

So by choosing $m > (1/2\pi)\sqrt{a_0/a_1}$ we get $\|q\| \to 0$ exponentially. Moreover, suppose $q_0 = 0, p_0 \in H_m$ then if $u = p + q$ is the solution of (7.1) with $u_0 = p_0$, we have $q(t) \equiv 0$ with $u(t) = p(t)$ where $p(t)$ solves

$$p_t = a_0 p + (a_1 + ib_1)p_{xx} + (a_3 + ib_3)\left(\int_0^1 |p|^2\, dx\right)p$$

$$p(0) = p_0.$$

Thus the linear space H_m is invariant under the solution. Consequently H_m is an Inertial manifold of dimension $2m + 1$ for any $m > (1/2\pi)\sqrt{a_0/a_1}$. On the other hand, to find a lower bound on the dimension of the global attractor we linearize (7.1) around the steady state solution $u_s \equiv 0$ to obtain the linear equation

$$v_1 = a_0 v + (a_1 + ib_1)Av. \tag{7.5}$$

Notice that $v(t) = A_k(t)\, e^{2\pi kxi}$ with $A_k(0) \neq 0$ is an unstable solution for (7.5) for every $|k| < (1/2\pi)\sqrt{a_0/a_1}$. Hence following Babin and Vishik (1992) (see also Temam (1988)), the dimension of the unstable manifold around $u_s \equiv 0$ is larger or equal to $2m_* + 1$ where $m_* = (1/2\pi)\sqrt{a_0/a_1}$, which also forms a lower bound on the dimension of the global attractor and of the Inertial Manifold. From the above, we conclude the following theorem:

Theorem 4. If $a_0 > 0$, $a_1 > 0$, $a_2 = b_2 = 0$ and $a_3 < 0$, then the linear space H_{m_*}, where $m_* = 2[(1/2\pi)\sqrt{a_0/a_1}] + 1$, is the Inertial Manifold of lowest dimension.

8. Conclusion and discussion

In this paper, we have discussed the effect of nonlocal interactions on the Ginzburg–Landau dynamics by studying a nonlocal Ginzburg–Landau type amplitude equation. We can see that, depending on the value and the sign of a_3, the nonlocal term, has the stabilizing/destabilizing effect. In fact, when $a_3 > -a_2 > 0$ the nonlocal term destabilizes the system too much that certain solutions blow up in finite time (see (2.3)). However if $a_2 + a_3 < 0$, we have global existence along with many other finer features such as the existence of global attractor in $L^2(0, 1)$ and the possession of inertial manifolds. We have in this case obtained the upper and lower bound for the dimension estimates for the global attractor and also the dimension of inertial manifold for the nonlocal amplitude equation (1.1). We further compare the estimates that we found with that of the corresponding cubic Ginzburg–Landau equation (1.4). We find that if $a_3 < 0$, then the dimension estimates that we found are smaller when compared to the cubic Ginzburg–Landau equation; which indicates a stabilizing effect of the nonlocal term in this case. On the other hand, when $a_3 > 0$ we find that the dimension estimates for the nonlocal equation is larger; which signifies destabilization due to the nonlocal term. It is worth mentioning that the above conclusion is based on global energy estimate around zero. For instance, we did not study here the effect of b_3 on the nature of the dynamics while say $a_3 = 0$. The special case $a_2 = b_2 = 0$ has been treated in section 7 and an explicit linear Inertial Manifold, which has a dimension equal to the dimension of the global attractor was found, hence it is of lowest dimension.

Acknowledgement

We would like to thank the anonymous referees for their helpful comments and suggestions. This work was partially supported by the ONR grant No. N00014-89-J-3023, the NSF grant No. DMS-9308774, and the University of California–Los Alamos National Laboratory INCOR program. Part of this work was done when E. S. Titi was visiting the Mettag–Leffler Institute in Sweden, and the Department of Mathematics at the Technion in Israel. E. S. Titi also acknowledges the support of the University of California–Irvine Faculty Research Fellowship.

References

[1] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam 1992.
[2] M. Bartuccelli, P. Constantin, C. R. Doering, J. D. Gibbon and M. Gisselfalt, *On the possibility of soft and hard turbulence in the complex Ginzburg–Landau equation*, Physica D *44*, 421–444 (1990).

[3] A. M. Bloch and E. S. Titi, *On the dynamics of the rotating elastic beams*, Proc. Conference on: New Trends in Systems Theory, A. Perdon and B. F. Wyman (eds), Birkhäuser, Boston (July 9–11, 1990).
[4] B. Cockburn, D. Jones and E. S. Titi, *Degrés de liberté déterminants pour équations nonlinéaires dissipatives*, Determining degrees of freedom for nonlinear dissipative equations, C.R. Acad. Sci.—Paris, Sér. I, *321*, 563–568 (1995).
[5] B. Cockburn, D. Jones and E. S. Titi, *Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems*, Preprint 1995.
[6] P. Collet, *Thermodynamic limit of the Ginzburg–Landau equations*, Nonlinearity *7*, 1175–1190 (1994).
[7] P. Constantin and C. Foias, *Navier–Stokes Equations*, The University of Chicago Press 1988.
[8] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, AMS *70*, Springer–Verlag 1989.
[9] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, *Spectral barriers and inertial manifolds for dissipative partial differential equations*, J. Dynamics & Diff. Eqns. *1*, 45–73 (1989).
[10] P. Constantin, C. Foias and R. Temam, *Attractor representing turbulent flows*, Memoirs of AMS *53*, No. 314 (1985).
[11] A. Debussche and R. Temam, *Inertial manifolds and their dimensions*, in *Dynamical Systems: Theory and Applications*, S. I. Anderson, A. E. Anderson and U. Ottoson (eds.), World Scientific, Singapore 1993.
[12] A. Doelman and E. S. Titi, *On the exponential rate of convergence of the Galerkin approximation in the Ginzburg–Landau equation*, Proc. NATO Advanced Research Workshop: Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters, M. Garbey and H. G. Kaper (eds.), Kluwer Academic Publ., Dordrecht 1993, pp. 241–252.
[13] A. Doelman and E. S. Titi, *Regularity of solutions and the convergence of the Galerkin method in the Ginzburg–Landau equation*, Num. Functional Analysis & Optimization *14*, 299–321 (1993).
[14] C. R. Doering, J. D. Gibbon, D. D. Holm and B. Nicolaenko, *Low dimensional behavior in the complex Ginzburg–Landau equation*, Nonlinearity *1*, 279–309 (1988).
[15] C. R. Doering, J. D. Gibbon and C. David Levermore, *Weak and strong solutions of the complex Ginzburg–Landau equation*, Physica D *71*, 285–318 (1994).
[16] J. Q. Duan, P. Holmes and E. S. Titi, *Regularity, approximation and asymptotic dynamics for a generalized Ginzburg-Landau equation*, Nonlinearity *6*, 915–933 (1993).
[17] F. J. Elmer, *Nonlinear and nonlocal dynamics of spatially extended systems: stationary states, bifurcations and stability*, Physica D *30*, 321–341 (1988).
[18] A. B. Ferrari and E. S. Titi, *Gevrey regularity of solutions of a class of analytic nonlinear parabolic equations*, Preprint 1994.
[19] C. Foias, G. Sell and R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, J. Diff. Eqns. *73*, 309–353 (1988).
[20] C. Foias, G. Sell and E. Titi, *Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations*, J. Dynamics & Diff. Eqns *1*, 199–244 (1989).
[21] C. Foias and R. Temam, *Some analytic and geometric properties of the solutions of the Navier–Stokes equations*, J. Math. Pures Appl. *58*, 339–368 (1979).
[22] C. Foias and R. Temam, *Determination of the solutions of the Navier–Stokes equations by a set of nodal values*, Math. Comput. *43*, 117–133 (1984).
[23] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier–Stokes equations*, J. Functional Anal *87*, 359–369 (1989).
[24] C. Foias and E. S. Titi, *Determining nodes, finite difference schemes and inertial manifolds*, Nonlinearity *4*, 135–153 (1991).
[25] J. M. Ghidaglia and B. Héron, *Dimensions of the attractors associated to the Ginzburg–Landau equation*, Physica D *28*, 282–304 (1987).
[26] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Math. Soc. 1988.
[27] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lectures Notes in Mathematics 840, Springer-Verlag, New York 1981.
[28] D. A. Jones and E. S. Titi, *On the number of determining nodes for the 2D Navier–Stokes equations*, J. Math. Anal. Appl. *168*, 72–88 (1992).
[29] D. A. Jones and E. S. Titi, *Determination of the solutions of the Navier–Stokes equations by finite volume elements*, Physica D *60*, 165–174 (1992).
[30] D. A. Jones and E. S. Titi, *Upper bounds on the number of determining modes, nodes, and volume elements for the Navier–Stokes equations*, Indiana Math. J. *42*, 875–887 (1993).
[31] I. Kukavica, *On the number of determining nodes for the Ginzburg–Landau equation*, Nonlinearity *5*, 997–1006 (1992).

[32] C. D. Levermore and M. Oliver, *The complex Ginzburg–Landau equation as a model problem*, Lectures in Applied Mathematics, AMS *31*, Providence, RI 1996, 141–190.
[33] B. J. Matkowsky and V. Volpert, *Coupled Ginzburg–Landau equations in gasless combustion*, Physica D *54*, 203–219 (1992).
[34] P. Metzener and M. R. E. Proctor, *Nonlocal effects on pattern selection in directional solidification*, Int. Conf. on Nonlinear Dynamics and Pattern Formation in Natural Environment, The Netherlands 1994.
[35] A. Mielke, *Reductions of PDEs on domains with several unbounded directions: a first step towards modulational equations*, J. Appl. Math. Phys. (ZAMP) *43*, 449–470 (1992).
[36] A. C. Newell, T. Passot and J. Lega, *Order parameter equations for patterns*, Ann. Rev. Fluid Mech. *25*, 399–453 (1993).
[37] M. Or-Guil, M. Bode and H.-G. Purwins, *Nonlocal modulation equations describing the interaction of Hopf and Turing bifurcations*, Int. Conf. on Nonlinear Dynamics and Pattern Formation in Natural Environment, The Netherlands 1994.
[38] D. T. Papageorgiou, C. Maldarelli and D. S. Rumschitzki, *Nonlinear interfacial stability of core-annular film flows*, Phys. Fluids *2*, 340–352 (1990).
[39] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin 1983.
[40] G. R. Sell and Y. C. You, *Inertial manifolds — The nonself-adjoint case*, J. Diff. Eqn. *96*, 203–255 (1992).
[41] P. Takáč, P. Bollerman, A. Doelman, A. van Harten and E. S. Titi, *Analyticity of essentially bounded solutions to strongly parabolic semilinear systems*, SIAM, J. Math. Anal. 1996 (to appear).
[42] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, Berlin 1988.
[43] J. M. Vega, *On the amplitude equations arising at the onset of the oscillatory instability in pattern formation*, SIAM, J. Math. Anal. *24*, 603–617 (1993).
[44] J. W. Wilder, B. F. Edwards, D. A. Vasquez and G. I. Sivashinsky, *Derivation of a nonlinear front evolution equation for chemical waves involving convection*, Physica D *73*, 217–226 (1994).

(Received: May 19, 1995; revised: October 5, 1995)