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# Nonlinear stability of one-layer geostrophic fronts ★

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## Abstract

In this paper, we study the nonlinear stability of one-layer geostrophic fronts described by frontal geostrophic equations. We show that the $\beta$-effect plays a crucial role in the stability of one-layer geostrophic fronts. Especially, we prove that the class of nonlinearly stable fronts is more restricted when the $\beta$-effect is present than when it is absent.

*Keywords:* Stability; Frontal dynamics; Hamiltonian dynamics; Geostrophic fronts; $\beta$-effect

## 1. Introduction

The complete mathematical analysis of the primitive equations governing geophysical (oceanic and atmospheric) systems is not possible at the present time. Simplified equations which describe only the essential aspects of large scale geostrophic motions have been developed. These simplified equations apply to different regimes, e.g., the quasi-geostrophic (QG) regime [9] and references therein), the intermediate geostrophic (IG) regime [2,13] the planetary geostrophic (PG) regime [12], and the frontal geostrophic (FG) regime [3]. The unification and generalization of these equations are discussed in [5,7]. The investigation of these simplified equations should provide some insight to the original geophysical systems governed by primitive equations.

In this paper, we consider the frontal geostrophic dynamics [3]. It is a reduced regime in a $1\frac{1}{2}$ layer shallow-water, reduced-gravity model (the upper layer is active while the lower layer is at rest) on the $\beta$ plane. A front refers to the interface between the two layers, or the intersection of the interface with the upper surface. The well-known quasi-geostrophic regime corresponds to motions for length scales at the order of the deformation radius with small interfacial displacements. The frontal geostrophic dynamics apply for finite interfacial displacements but with length scales large compared to the deformation radius. Both regimes assume that the Rossby number $\epsilon$ and the $\beta$-effect are small

$$\epsilon \ll 1, \qquad \beta \ll 1.$$

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Therefore in both regimes, inertial oscillations are eliminated and the meridional contribution to the Coriolis force $f = f_0 + \beta y$ is small. Moreover, frontal geostrophic motions evolve much slower than quasi-geostrophic motions. With these scalings, the frontal geostrophic dynamics yield a single partial differential equation for the upper layer depth $h(x, y, t)$, i.e., the location of the interface. Depending on the relative size of the Rossby number $\epsilon$ and the $\beta$-effect, the partial differential equation for $h$ may take three different forms [3]:

(i) When $\beta \sim \epsilon$, the equation is

$$h_t - J(h, h \triangle h + \tfrac{1}{2} \nabla h \cdot \nabla h + \alpha y h) = 0, \tag{1.1}$$

where $\alpha = \beta/\epsilon$, and $J(f, g) = f_x g_y - f_y g_x$ is the Jacobian operator.

(ii) When the $\beta$-effect is neglected, the equation is

$$h_t - J(h, h \triangle h + \tfrac{1}{2} \nabla h \cdot \nabla h) = 0. \tag{1.2}$$

(iii) When $\beta \gg \epsilon$, the equation is

$$h_t - h h_x = 0. \tag{1.3}$$

Eqs. (1.1) and (1.3) were also derived by Williams and Yamagata [12]. The ratio $\omega$ of inertial period to motion time scale is at the order much larger than $\epsilon^2$ in the case (iii) above, while for the cases (i) and (ii), the ratio $\omega$ is at the order $\epsilon^2$.

We consider the cases where the interfaces intersect with the surface of the top layer. A line along which $h(x, y, t) = 0$ is called an outcrop or front, which separates two different regions in geophysical fluids. In this paper, we are interested in cases with closed outcrops. Example for such cases are Gulf Stream cyclonic rings formed by Gulf Stream meanders (e.g., [10]) It is important to understand the evolution and stability of fronts. In this paper, we study the formal and nonlinear stability of steady states (i.e., steady fronts) of Eqs. (1.1–1.3). Especially, we will check the role of the $\beta$-effect in the stability conditions.

## 2. Hamiltonian formalism

We use $\mathcal{R}$ to denote the two-dimensional bounded domain on which the frontal geostrophic motion occurs. We denote $\Omega \subset \mathcal{R}$ as the two-dimensional domain where $h > 0$. The boundary of $\Omega$ is an outcrop or front. In the following, all double integrals are on the domain $\Omega$ unless specified otherwise.

To use the Arnold method to study the stability of fronts, as in [11], we write Eq. (1.1) or (1.2) as a Hamiltonian system [8]

$$h_t = \mathcal{D}\frac{\delta H}{\delta h}, \tag{2.1}$$

where

$$H(h) = -\frac{1}{2}\int\int h\nabla h \cdot \nabla h \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{2}\int\int \alpha y h^2 \, \mathrm{d}x \, \mathrm{d}y, \tag{2.2}$$

and $\mathcal{D}$ is a skew-symmetric operator defined by

$$\mathcal{D}\cdot = J(h, \cdot). \tag{2.3}$$

In the Arnold method we need to find more constants of motion (so-called Casimir functions). This can be easily done using a Poisson bracket

$$[H_1, H_2] = \left( \frac{\delta H_1}{\delta h}, \mathcal{D}\frac{\delta H_2}{\delta h} \right), \tag{2.4}$$

where $(\cdot, \cdot)$ is the usual scalar product in $L^2(\Omega)$. We can also rewrite Eq. (1.1) as

$$h_t = [h, H], \tag{2.5}$$

as long as we think $h$ as

$$h = \iint \delta(x - x')\delta(y - y')h(x', y', t)\, \mathrm{d}x'\, \mathrm{d}y'. \tag{2.6}$$

The Casimir functions $C(h)$ can be found by solving

$$0 = [F, C(h)] = \left( \frac{\delta F}{\delta h}, \mathcal{D}\frac{\delta C}{\delta h} \right) \tag{2.7}$$

for all $F(h) \in L^2(\Omega)$. Hence we have

$$J(h, \delta C/\delta h) = 0, \tag{2.8}$$

which means that $\delta C/\delta h$ needs to be an arbitrary (smooth) function of $h$, or

$$C(h) = \iint g(h)\, \mathrm{d}x\, \mathrm{d}y \tag{2.9}$$

for arbitrary function $g(h)$.

The steady states $h_0(x, y)$ of Eq. (1.1) satisfy

$$J(h_0, h_0 \bigtriangleup h_0 + \tfrac{1}{2} \nabla h_0 \cdot \nabla h_0 + \alpha y h_0) = 0, \tag{2.10}$$

or

$$h_0 \bigtriangleup h_0 + \tfrac{1}{2} \nabla h_0 \cdot \nabla h_0 + \alpha y h_0 = f(h_0) \tag{2.11}$$

for arbitrary (smooth) single-valued function $f(h)$. In particular, all zonal steady flow $h_0(y)$ are steady states for Eq. (1.1).

For Eq. (1.3), we can also write it in Hamiltonian form as above with

$$H(h) = \frac{1}{2} \int \int y h^2\, \mathrm{d}x\, \mathrm{d}y. \tag{2.12}$$

The Casimir functions are still as in (2.9) and all steady states satisfy

$$J(h_0, y h_0) = 0, \tag{2.13}$$

or

$$y h_0 = f(h_0) \tag{2.14}$$

for arbitrary (smooth) function $f(h)$.

## 3. Formal and linear stability

To study formal stability of a steady state $h_0(x, y)$ of Eq. (1.1), we need to show that there exists a constant of motion $\tilde{H}$ such that $\delta\tilde{H}(h_0) = 0$ and $\delta^2\tilde{H}(h_0)$ is sign-definite [6].

We consider the constrained Hamiltonian function

$$\tilde{H}(h) = H(h) + C(h), \tag{3.1}$$

with the Casimir $C(h)$ satisfying

$$0 = \delta\tilde{H}(h_0) = \int\int (h_0\Delta h_0 + \tfrac{1}{2}\nabla h_0 \cdot \nabla h_0 + \alpha y h_0 + g'(h_0))\delta h \, \mathrm{d}x \, \mathrm{d}y, \tag{3.2}$$

i.e.,

$$g(h) = -\int^{h} f(\xi) \, \mathrm{d}\xi, \tag{3.3}$$

$$C(h) = -\int\int \left( \int^{h} f(\xi) \, \mathrm{d}\xi \right) \mathrm{d}x \, \mathrm{d}y, \tag{3.4}$$

where $f(\cdot)$ defines the steady state $h_0$ via (2.11). With this chosen $C(h)$, we calculate

$$\delta^2\tilde{H}(h_0) = \int\int [(\Delta h_0 + \alpha y - f'(h_0))(\delta h)^2 - h_0\nabla(\delta h) \cdot \nabla(\delta h)] \, \mathrm{d}x \, \mathrm{d}y. \tag{3.5}$$

We note that $f'(h_0)$ can be obtained by differentiating Eq. (2.11). Hence we have

$$\delta^2\tilde{H}(h_0) = -\int\int h_0 h_{0y}^2 \nabla(\delta h/h_{0y}) \cdot \nabla(\delta h/h_{0y}) \, \mathrm{d}x \, \mathrm{d}y - \alpha \int\int \frac{h_0}{h_{0y}}(\delta h)^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{3.6}$$

We know that $h_0 > 0$ on the domain $\Omega$. Therefore, for meridionally increasing fronts ($h_{0y} > 0$), $\delta^2\tilde{H}(h_0)$ is negative-definite. Hence we have the following result.

*Theorem 1.*

(i) When the $\beta$-effect is at the order of the Rossby number $\epsilon$, all meridionally increasing steady fronts are formally stable, and hence it is linearly stable in the sense of Lyapunov in the norm $||h|| = \sqrt{-\delta^2\tilde{H}(h)}$.

(ii) When the $\beta$-effect is zero, all steady fronts are formally and hence linearly stable in the sense of Lyapunov.

We remark that the stability condition in (i) above may not be sharp, and that the conclusion in (ii) above was proved in [11] and can be easily seen from (3.6). The statement (i) in Theorem 1 also implies the result of Benilov [1] who showed that when the $\beta$-effect is present, all meridionally increasing steady fronts are spectrally stable (no positive eigenvalue). Our result is generally stronger than Benilov's result.

For Eq. (1.3), we also consider a constrained Hamiltonian function as in (3.1) but with $H(h)$ defined by (2.12). We choose $C(h)$ such that

$$0 = \delta\tilde{H}(h_0) = \int\int [y h_0 + g'(h_0)]\delta h \, \mathrm{d}x \, \mathrm{d}y \tag{3.7}$$

or

$$g(h) = -\int^{h} f(\xi)\,\mathrm{d}\xi, \tag{3.8}$$

$$C(h) = -\int\int\left(\int^{h} f(\xi)\,\mathrm{d}\xi\right)\mathrm{d}x\,\mathrm{d}y. \tag{3.9}$$

We calculate

$$\delta^2\tilde{H}(h_0) = \int\int [y - g'(h_0)](\delta h)^2\,\mathrm{d}x\,\mathrm{d}y = -\int\int \frac{h_0}{h_{0y}}(\delta h)^2\,\mathrm{d}x\,\mathrm{d}y, \tag{3.10}$$

which is negative-definite for meridionally increasing steady fronts ($h_{0y} > 0$), and positive-definite for meridionally decreasing steady fronts ($h_{0y} < 0$).

Hence we have the following result.

*Theorem 2.* When the $\beta$-effect is much larger than the Rossby number $\epsilon$, both meridionally increasing and decreasing steady fronts are formally stable, and hence linearly stable in the sense of Lyapunov, in the norm $\|h\| = \sqrt{-\delta^2\tilde{H}(h)}$ or $\|h\| = \sqrt{\delta^2\tilde{H}(h)}$, respectively.

## 4. Nonlinear stability

To show nonlinear stability of steady fronts, we need to derive global bounds (in some norm) of the disturbances in terms of known data. To this end we consider the conserved functional as in [11]:

$$\mathcal{L}(h) = H(h_0 + h) - H(h_0) + C(h_0 + h) - C(h_0), \tag{4.1}$$

which becomes, after using (2.2), (3.4) and (2.11),

$$\begin{aligned}\mathcal{L}(h) = {} & \frac{1}{2}\int\int [\Delta h_0 h^2 - (h_0 + h)\nabla h \cdot \nabla h + \alpha y(2h_0 h + h^2)]\,\mathrm{d}x\,\mathrm{d}y \\ & + \int\int \left[\int_{h_0+h}^{h_0} f(\xi)\,\mathrm{d}\xi + f(h_0)h\right]\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.2}$$

We assume that

$$a_1 < f'(\xi) < b_1, \tag{4.3}$$

$$1 + \sup_{\mathcal{R}}(\Delta h_0 + \alpha y) < a_1, \tag{4.4}$$

where $a_1, b_1$ are finite real numbers. If condition (4.3) is integrated twice, it follows that

$$-\tfrac{1}{2}b_1 h^2 < \int_{h_0+h}^{h_0} f(\xi)\,\mathrm{d}\xi + f(h_0)h < -\tfrac{1}{2}a_1 h^2. \tag{4.5}$$

Using (4.5) and the fact $2\alpha y h_0 h < (\alpha y h_0)^2 + h^2$ in (4.2), we obtain

$$\begin{aligned} 2\mathcal{L}(h) &< \int\!\!\int [(\Delta h_0 + \alpha y - a_1)h^2 - (h_0 + h)\nabla h \cdot \nabla h + 2\alpha y h_0 h]\,\mathrm{d}x\,\mathrm{d}y \\ &< \int\!\!\int [(\Delta h_0 + \alpha y - a_1 + 1)h^2 - (h_0 + h)\nabla h \cdot \nabla h + (\alpha y h_0)^2]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.6}$$

Moreover, we denote $\hat{h}(x, y) = h(x, y, 0)$, and by using (4.5) and the fact $2\alpha y h_0 h > -(\alpha y h_0)^2 - h^2$ in (4.2), we obtain

$$\begin{aligned} 2\mathcal{L}(\hat{h}) &> \int\!\!\int [(\Delta h_0 + \alpha y - b_1)\hat{h}^2 - (h_0 + \hat{h})\nabla \hat{h} \cdot \nabla \hat{h} + 2\alpha y h_0 \hat{h}]\,\mathrm{d}x\,\mathrm{d}y \\ &> \int\!\!\int [(\Delta h_0 + \alpha y - b_1 - 1)\hat{h}^2 - (h_0 + \hat{h})\nabla \hat{h} \cdot \nabla \hat{h} - (\alpha y h_0)^2]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.7}$$

We note that $\mathcal{L}(h) = \mathcal{L}(\hat{h})$, since $\mathcal{L}$ is a conserved functional. Combining this fact with (4.6) and (4.7), we get

$$\begin{aligned} 0 &< \int\!\!\int [(a_1 - 1 - \Delta h_0 - \alpha y)h^2 + (h_0 + h)\nabla h \cdot \nabla h]\,\mathrm{d}x\,\mathrm{d}y \\ &< \int\!\!\int [(1 + b_1 - \Delta h_0 - \alpha y)\hat{h}^2 + (h_0 + \hat{h})\nabla \hat{h} \cdot \nabla \hat{h} + 2(\alpha y h_0)^2]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.8}$$

We have used the fact that $(h_0 + h)\nabla h \cdot \nabla h \geq 0$ since the total thickness must satisfy $h_0 + h \geq 0$. From (4.8) we further obtain

$$0 < E(h) < \Gamma E(\hat{h}), \tag{4.9}$$

where

$$E(h) \equiv \int\!\!\int [h^2 + (h_0 + h)\nabla h \cdot \nabla h]\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$\Gamma \equiv \frac{\max(\sup_{\mathcal{R}}(1 + b_1 - \Delta h_0 - \alpha y), 1)}{\min(\inf_{\mathcal{R}}(a_1 - 1 - \Delta h_0 - \alpha y), 1)}.$$

Note that the right-hand side of (4.9) is nonnegative and depends only on known data. This shows the nonlinear stability of $h_0(x, y)$ in the norm $E(h)$. The estimate (4.9) also implies that $\int\!\int h^2\,\mathrm{d}x\,\mathrm{d}y$ is bounded by known data. So $h_0(x, y)$ is also nonlinearly stable in the norm $\int\!\int h^2\,\mathrm{d}x\,\mathrm{d}y$. Hence we have the following result.

*Theorem 3.* Let $h_0(x, y)$ be a steady front for Eq. (1.1), i.e., it satisfies

$$h_0 \Delta h_0 + \tfrac{1}{2} \nabla h_0 \cdot \nabla h_0 + \alpha y h_0 = f(h_0) \tag{4.10}$$

for an arbitrary (smooth) single-valued function $f(h)$. If $f(h)$ satisfies the following conditions

$$a_1 < f'(\xi) < b_1, \tag{4.11}$$

$$1 + \sup_{\mathcal{R}}(\Delta h_0 + \alpha y) < a_1, \tag{4.12}$$

for some finite real numbers $a_1, b_1$, then, $h_0(x, y)$ is nonlinearly stable in the norm $\int\!\int h^2\,\mathrm{d}x\,\mathrm{d}y$, and in the norm $E(h) \equiv \int\!\int [h^2 + (h_0 + h)\nabla h \cdot \nabla h]\,\mathrm{d}x\,\mathrm{d}y$.

We remark that, if the $\beta$-effect is not zero, i.e., $\alpha = \beta/\epsilon \neq 0$, then the nonlinear stability condition (4.12) directly involves the meridional stretch or size of the frontal region. When $\beta = 0$, the stability condition only involves $h_0$ itself. This indicates that the set of stable fronts is generally smaller when $\beta \neq 0$.

Now we consider Eq. (1.3). We follow the scheme in [6]. From (2.12) and (3.9), we calculate

$$H(h_0+h) - H(h_0) - DH(h_0)\cdot h = \frac{1}{2}\int\int [y(2h_0 h + h^2) - 2yh_0 h] = \frac{1}{2}\int\int yh^2 \equiv Q_1(h), \tag{4.13}$$

$$C(h_0+h) - C(h_0) - DC(h_0)\cdot h = \int\int \left[\int_{h_0+h}^{h_0} f(\xi)\,\mathrm{d}\xi + f(h_0)h\right]. \tag{4.14}$$

We assume that

$$a_2 < f'(\xi) < b_2 \quad \text{on } \mathcal{R} \tag{4.15}$$

for some real finite numbers $a_2$ and $b_2$. Then from (4.14) we have

$$C(h_0+h) - C(h_0) - DC(h_0)\cdot h > -\frac{1}{2}b_2\int\int h^2 \equiv \hat{Q}_2(h) \tag{4.16}$$

or

$$C(h_0+h) - C(h_0) - DC(h_0)\cdot h < -\frac{1}{2}a_2\int\int h^2 \equiv \check{Q}_2(h). \tag{4.17}$$

From Holm et al. [6], the nonlinear stability of $h_0(x, y)$ is ensured by either

$$Q_1(h) + \hat{Q}_2(h) = \int\int (y - b_2)h^2 > 0, \tag{4.18}$$

i.e.,

$$\inf_{\mathcal{R}}(y) > b_2, \tag{4.19}$$

or

$$Q_1(h) + \check{Q}_2(h) = \int\int (y - a_2)h^2 < 0, \tag{4.20}$$

i.e.,

$$\sup_{\mathcal{R}}(y) < a_2. \tag{4.21}$$

Thus we have proved the following conclusion.

*Theorem 4.* Let $h_0(x, y)$ be a steady solution of Eq. (1.3), i.e.,

$$yh_0 = f(h_0) \tag{4.22}$$

for an arbitrary (smooth) function $f(h)$. If $f(h)$ satisfies

$$a_2 < f'(\xi) < b_2 \quad \text{on } \mathcal{R} \tag{4.23}$$

for some real finite numbers $a_2$ and $b_2$, and if either

$$\inf_{\mathcal{R}}(y) > b_2, \tag{4.24}$$

or

$$\sup_{\mathcal{R}}(y) < a_2 \tag{4.25}$$

then $h_0(x, y)$ is nonlinearly stable under the norm $\|h\| = \sqrt{Q_1(h) + \hat{Q}_2(h)}$ or $\|h\| = \sqrt{-Q_1(h) - \check{Q}_2(h)}$, respectively.

We remark that, when the $\beta$-effect is large comparing with the Rossby number $\epsilon$, the nonlinear stability condition also depends on the meridional stretch or size of the frontal region (Theorem 4).

## 5. Discussions

In this paper, we have studied the formal and nonlinear stability of one-layer geostrophic fronts. We have seen that when the $\beta$-effect is present, the class of formally and linearly stable fronts is more restricted than when the $\beta$-effect is absent. When the $\beta$-effect is present, the nonlinear stability condition involves with the meridional stretch or size of the frontal region.

We should remark that the models used here do not include stratification or baroclinicity. The result may not directly be applied to real geophysical fronts (such as Gulf Stream rings) without the analysis of the effect of baroclinicity.

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