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PROBABILISTIC DYNAMICS OF TWO-LAYER GEOPHYSICAL FLOWS

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The two-layer quasigeostrophic flow model is an intermediate system between the single-layer two-dimensional barotropic flow model and the continuously stratified three-dimensional baroclinic flow model. This model is widely used to investigate basic mechanisms in geophysical flows, such as baroclinic effects, the Gulf stream and subtropical gyres.

We consider the two-layer quasigeostrophic flow model under stochastic wind forcing on the top layer. The fluctuating part of the wind forcing is modeled as the generalized time derivative of a Wiener process. We first transform this stochastic two-layer fluid system into a coupled system of random partial differential equations. Then we prove that the stochastic two-layer fluid system has finite sets of asymptotically determining functionals (such as determining modes and determining nodes) in probability. Furthermore, we show that the asymptotic probabilistic dynamics of this system depends only on the top fluid layer. Namely, in the probability sense and asymptotically, the dynamics of the two-layer quasigeostrophic fluid system is determined by the top layer. In other words, the bottom layer is slaved by the top layer. This conclusion is true provided that the Wiener process and the fluid parameters satisfy a certain condition. In particular, this latter condition is satisfied when the trace of the covariance operator of the Wiener process is small enough and the Ekman constant r is sufficiently large.

Keywords: Stochastic geophysical flow models; determining functionals for random systems; random dynamical systems; random wind forcing.

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1. Introduction

The continuously stratified, three-dimensional (3D) baroclinic quasigeostrophic flow model describes large scale geophysical fluid motions in the atmosphere and oceans. This model is much simpler than the primitive flow model or the rotating Navier–Stokes flow model. When the fluid density is approximately constant, this model reduces to the barotropic, single-layer, two-dimensional (2D) quasigeostrophic model. The two-layer quasigeostrophic flow model, in which the fluid consists of two homogeneous fluid layers of uniform but distinct densities ρ_1 and ρ_2, is an intermediate system between the single-layer 2D barotropic flow model and the continuously stratified, 3D baroclinic flow model.

The two-layer quasigeostrophic flow model has been used as a theoretical and numerical model to understand basic mechanisms in large scale geophysical flows, such as baroclinic effects [32], wind-driven circulation [5], [4], the Gulf Stream [26], fluid stability [3] and subtropical gyres [33], [31]. Recently Salmon [35] introduced a generalized two-layer ocean flow model.

We consider the two-layer quasigeostrophic flow model ([32], p. 423):

$$\begin{aligned} \frac{\partial q_1}{\partial t} + J(\psi_1, q_1 + \beta y) &= \nu \Delta^2 \psi_1 + f + \dot{W}\,, \\ \frac{\partial q_2}{\partial t} + J(\psi_2, q_2 + \beta y) &= \nu \Delta^2 \psi_2 - r \Delta \psi_2\,, \end{aligned} \tag{1}$$

where potential vorticities $q_1(x, y, t)$, $q_2(x, y, t)$ for the top layer and the bottom layer are defined via stream functions $\psi_1(x, y, t)$, $\psi_2(x, y, t)$, respectively,

$$\begin{aligned} q_1 &= \Delta\psi_1 - F_1 \cdot (\psi_1 - \psi_2)\,, \\ q_2 &= \Delta\psi_2 - F_2 \cdot (\psi_2 - \psi_1)\,. \end{aligned} \tag{2}$$

Here x, y are Cartesian coordinates in zonal (east), meridional (north) directions, respectively; $(x, y) \in O := (0, L) \times (0, L)$, where L is a positive number; F_1, F_2 are positive constants defined by (see also [36], p. 87)

$$F_1 = \frac{f_0^2}{gh_1} \frac{\rho_0}{\rho_2 - \rho_1}\,,$$

$$F_2 = \frac{f_0^2}{gh_2} \frac{\rho_0}{\rho_2 - \rho_1}\,,$$

with g the gravitational acceleration; h_1, h_2 the depth of top and bottom layers, ρ_1, ρ_2 the densities ($\rho_2 > \rho_1$) of top and bottom layers, respectively; and L, ρ_0 the characteristic scales for horizontal length and density of the flows, respectively; $f_0 + \beta y$ (with f_0, β constants) is the Coriolis parameter and β is the meridional gradient of the Coriolis parameter; $\nu > 0$ is the viscosity. Note that $r = f_0 \frac{\delta_{\mathrm{E}}}{2(h_1+h_2)}$ is the Ekman constant ([33], p. 29) which measures the intensity of friction at the bottom boundary layer (the so-called Ekman layer) or the rate for vorticity decay due to the friction in the Ekman layer. Here $\delta_{\mathrm{E}} = \sqrt{2\nu/f_0}$ is the Ekman layer thickness ([32], p. 188). Moreover, $J(h, g) = h_x g_y - h_y g_x$ is the Jacobi operator and

$\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator. Finally, $f(x, y, t)$ is the mean (deterministic) wind forcing with average zero: $\int_O f dO = 0$.

An important part of the above flow model (1) is the white noise term $\dot{W}$, which is the generalized derivative of a Wiener process $W(t)$ with respect to time t, in an appropriate function space to be specified below. This white noise term describes the fluctuating part of the external wind forcing in the top fluid layer; see Hasselmann [24] and Arnold [2]. The fluctuating part is usually of a shorter time scale than the response time scale of the large scale quasigeostrophic flows. So we neglect the autocorrelation time of this fluctuating process. We thus assume the noise is white in time but it is allowed to be colored in space, i.e. it may be correlated in space variables x and y; see [34]. In the quasigeostrophic fluid regime, the forcing term $f + \dot{W}$ is the wind stress curl [32] and it is a major source of relative vorticity. The Wiener process (also a Gaussian process) $W(t)$ has zero mean and is characterized by its covariance operator Q. There has been some analysis on wind stress curl data from the National Aeronautics and Space Administration Scatterometer (NSCAT) and from the National Center for Environmental Prediction (NCEP); see, for example, [29], [10]. Such data analysis also involves estimating the covariance and its trace, and the trace is usually taken to be finite. In this paper, we consider the case when the covariance operator Q of the Wiener process has a finite trace.

We assume periodic boundary conditions for ψ in x and y with period L. In addition, we require that

$$\int_O \psi dO = 0\,.$$

We also assume an appropriate initial condition

$$q(x, y, 0) = q_0(x, y)\,.$$

The stochastically forced quasigeostrophic model has been used to investigate various phenomena in geophysical flows [25], [30], [37], [23], [17], [9].

The two-layer quasigeostrophic flow model (1) is a coupled system of stochastic *and* deterministic partial differential equations. This coupling is via *algebraic* relations (2). We cannot directly apply random dynamical systems methods [1] to this stochastic system. Therefore, we will transform this system into a coupled system of partial differential equations with random coefficients but without white noise explicitly appearing in the equations. We can achieve this by a random coordinate transformation. The solution of this transformed system of random partial differential equations then generates a random dynamical system and thus we can use techniques from the theory of random dynamical systems. In order to understand the long time dynamics, we will apply the method of *determining functionals* in a version suitable for random dynamical systems which is based on convergence in probability; see Chueshov *et al.* [14]. We will show that the asymptotic long time behavior is determined by the asymptotic behavior of finitely many functionals. For instance, we can choose these linear functionals as the Fourier modes given by the eigenfunctions of the Laplace operator. Furthermore, we show that certain linear functionals defined on the top fluid layer alone can *determine* the asymptotical behavior of the complete two-layer system, when the fluid parameters and the Wiener process satisfy a certain condition, such that the Ekman constant r is sufficiently large and the trace of the covariance operator of the Wiener process is small enough. Note that the generalized time derivative of the Wiener process models the fluctuating part of the wind stress forcing on the top fluid layer, and the Ekman constant r measures the rate of vorticity decay due to the friction in the bottom Ekman layer.

In this paper, we first recall some basic facts about random dynamical systems in Sec. 2. In Sec. 3, we establish the well-posedness of the stochastic two-layer quasigeostrophic model by transforming it into a coupled system of random partial differential equations. The main results on asymptotic probabilistic determining functionals are presented in Secs. 4–6. We then present some examples in Sec. 7. Finally we summarize our conclusions in Sec. 8.

2. Random Dynamical System and Determining Functionals

In order to investigate the long time dynamics of the two-layer fluid system (1) under the influence of random forces, we need some appropriate concepts and tools from the theory of *random dynamical systems*.

A random dynamical system consists of two components. The first component is a *metric dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ as a model for a noise, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F}$ measurable flow: we have

$$\theta_0 = \mathrm{id}\,, \quad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t\theta_\tau$$

for t, $\tau \in \mathbb{R}$. To express that the noise is stationary and chaotic, the measure $\mathbb{P}$ is supposed to be ergodic with respect to θ. The second component of a random dynamical system is a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H)$-measurable mapping φ satisfying the *cocycle* property

$$\varphi(t+\tau, \omega, x) = \varphi(t, \theta_\tau\omega, \varphi(\tau, \omega, x))\,, \quad \varphi(0, \omega, x) = x\,,$$

where the phase space H is a separable metric space and x is chosen arbitrarily in H. We will denote this random dynamical system by symbol φ.

A standard model for such a noise θ is the two-sided *Brownian motion*: Let U be a separable Hilbert space. We consider the probability space

$$(C_0(\mathbb{R}, U), \mathcal{B}(C_0(\mathbb{R}, U)), \mathbb{P})\,,$$

where $C_0(\mathbb{R}, U)$ is the Fréchet space of continuous functions on $\mathbb{R}$ of uniform convergence on compact intervals which are zero at zero and $\mathcal{B}(C_0(\mathbb{R}, U))$ is the corresponding Borel σ-algebra. Suppose that we have a covariance operator Q on U. Then $\mathbb{P}$ denotes the *Wiener measure* with respect to Q. Note that $\mathbb{P}$ is ergodic with respect to the *Wiener shift* flow θ_t:

$$\theta_t\omega = \omega(\cdot + t) - \omega(t)\,, \quad \text{for} \quad \omega \in C_0(\mathbb{R}, U)\,. \tag{3}$$

A major example of a random dynamical system is a random differential equation. For example, let us consider the following evolution equation in some Hilbert space

$$\frac{du}{dt} = F(u, \theta_t\omega)\,, \quad u(0) = x, \tag{4}$$

over some metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. If (4) is well-posed for every $\omega \in \Omega$ and solutions $u(t, \omega; x)$ depend measurably on (t, ω, x), then the operator

$$\varphi\,:\,(t, \omega, x) \to u(t, \omega; x)$$

defines a random dynamical system (cocycle) φ. For detailed presentation of random dynamical systems we refer to the monograph by Arnold [1].

Motivated by deterministic dynamical systems we introduce several useful concepts from the theory of random dynamical systems.

A closed set $B(\omega)$, depending on ω, in a separable Hilbert space H is called random if the distance mapping $\omega \to \sup_{x\in B(\omega)} \|x - y\|_H$ is a random variable for any $y \in H$.

A random dynamical system is called *dissipative* if there exists a random set B that is bounded for any ω and that is absorbing: for any random variable $x(\omega) \in H$ there exists a $t_x(\omega) > 0$ such that if $t \geq t_x(\omega)$, then

$$\varphi(t, \omega, x(\omega)) \in B(\theta_t\omega).$$

In the deterministic case (φ is independent of ω) the last relation coincides with the definition of an absorbing set. In the case of partial differential equations of *parabolic type*, due to the smoothing property, it is usually possible to prove that a dissipative system possesses compact invariant absorbing sets. For more details, see Temam [38], p. 22f. Hence for a system of stochastically forced *parabolic* partial differential equations, such as the stochastic two-layer fluid system introduced in the last section, we usually consider the random set $B(\omega)$ to be compact. In addition, we will assume that $B(\omega)$ is forward invariant:

$$\varphi(t, \omega, B(\omega)) \subset B(\theta_t\omega)\,, \quad t > 0\,.$$

In the following we also need a concept of *tempered random variables.* A random variable x is called tempered if

$$t \to |x(\theta_t\omega)|$$

is subexponentially growing:

$$\limsup_{t\to\pm\infty} \frac{\log^+ |x(\theta_t\omega)|}{|t|} = 0 \quad \text{a.s.}$$

This technical condition is not a very strong restriction because the only alternative is that the above lim sup is ∞, which describes the degenerate case of stationarity; see Arnold [1], p. 164f.

Our main purpose is to estimate the degree of freedom of the long time dynamics of a random dynamical system arising from the two-layer oceanic flow problem introduced above. We will apply the theory of determining functionals to estimate the degree of freedom. Since the dynamics of this model is influenced by random forces, we will apply the theory of determining functionals in the context of convergence in probability.

We now introduce our basic definition.

Definition 2.1. Let V be a Banach space which is continuously embedded in H. Assume that there exists $\tau > 0$ such that $\varphi(\cdot, \omega, x) \in L^2_{\text{loc}}(\tau, +\infty; V)$ for almost all $\omega \in \Omega$ and $x \in H$. A set $\mathcal{L} = \{l_j,\ j = 1, \dots, N\}$ of linear continuous and linearly independent functionals on V is called asymptotically determining in probability if

$$(\mathbb{P}) \lim_{t\to\infty} \int_t^{t+1} \max_j |l_j(\varphi(\tau, \omega, x_1(\omega)) - \varphi(\tau, \omega, x_2(\omega)))|^2 d\tau = 0$$

for two initial conditions $x_1(\omega)$, $x_2(\omega) \in H$ implies

$$(\mathbb{P}) \lim_{t\to\infty} \|\varphi(t, \omega, x_1(\omega)) - \varphi(t, \omega, x_2(\omega))\|_H = 0\,.$$

Note that $L^2_{\text{loc}}(\tau, +\infty; V)$ is the space of all functions $\varphi(\cdot)$ such that $\varphi(\cdot) \in L^2(\tau, n, V)$ for any $n > \tau$. The theory of determining functionals was started with the papers [21] and [28] devoted to 2D Navier–Stokes equations. This theory is well-developed for deterministic systems (see, e.g., [20], [22], [27], [7], [12], [11] and the references therein). Some results are also available for stochastic systems (see, e.g., [8], [13], [14], [19]). One of the main advantages of this theory is the possibility of locating spatial domains or parameters which are responsible for the long-time dynamics. The existence of a finite number of determining functionals means that the long-time behavior of the system is finite-dimensional. Moreover the values of these functionals on solutions can be interpreted as a result of some measurement of the system. From an applied point of view the finiteness of number of determining functionals means that we need only a finite number of devices to observe completely all of the dynamics of the system.

Before we formulate the main theorem for determining functionals, let us explain the idea by a simple deterministic example.

We consider a deterministic partial differential equation

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(u)\,, \quad u(0, x) = u_0(x)\,. \tag{5}$$

For simplicity we assume that for $x \in O$ (see Sec. 1) the solution $u(t, x)$ satisfies periodic boundary conditions. In addition, we assume that ν is a positive constant and the function $f(u)$ is Lipschitz continuous with Lipschitz constant C_{Lip}:

$$|f(u_1) - f(u_2)| \le C_{\text{Lip}} |u_1 - u_2| \quad \text{for} \quad u_1,\, u_2 \in \mathbb{R}\,.$$

We now need an embedding inequality: There exists a positive constant ε_0 such that:

$$\|u\|_H \le \varepsilon_0 \|\nabla u\|_H \quad \text{for} \quad \nabla u \in H. \tag{6}$$

Suppose for the moment that ν is *large* and C_{Lip} is sufficiently small that

$$C_{\mathrm{Lip}} - \varepsilon_0^{-2}\nu < 0\,. \tag{7}$$

We now study the difference of two solutions of Eq. (5):

$$\frac{\partial(u_1 - u_2)}{\partial t} = \nu\Delta(u_1 - u_2) + f(u_1) - f(u_2)\,,$$
$$u_1(0,x) = u_0^1(x) \in H\,, \quad u_2(0,x) = u_0^2(x) \in H\,.$$

Using the chain rule and the above embedding inequality we arrive at

$$\begin{aligned}\frac{d}{dt}\|u_1(t) - u_2(t)\|_H^2 &\le -2\nu\|\nabla(u_1(t) - u_2(t))\|_H^2 + 2C_{Lip}\|u_1(t) - u_2(t)\|_H^2 \\ &\le -2\nu\varepsilon_0^{-2}\|u_1(t) - u_2(t)\|_H^2 + 2C_{Lip}\|u_1(t) - u_2(t)\|_H^2\,. \end{aligned} \tag{8}$$

It can be verified that by the Gronwall inequality

$$\lim_{t\to\infty} \|u_1(t) - u_2(t)\|_H = 0\,.$$

In particular we can deduce that there exists an exponentially attracting steady state. The asymptotic dynamics is restricted to one point, the steady state. Hence the asymptotic dynamics is zero dimensional. No determining functionals are needed to justify that any two trajectories approach each other.

Suppose now that ν is not large or C_{Lip} is not small, so that (7) is not satisfied. In this case we cannot use the embedding inequality (6). In particular, we have to use another kind of inequality which could be called an *additive embedding inequality.* This inequality is based on a finite set of linear continuous functionals $\mathcal{L} = \{l_1, \ldots, l_N\}$ defined on the set of functions on O such that $\nabla u \in H$:

$$\|u\|_H \le C_{\mathcal{L}} \max_{l_i\in\mathcal{L}} |l_i(u)| + \varepsilon_{\mathcal{L}}\|\nabla u\|_H\,, \tag{9}$$

where $C_{\mathcal{L}}$, $\varepsilon_{\mathcal{L}}$ are some positive constants depending on $\mathcal{L}$. It is crucial to control the constant $\varepsilon_{\mathcal{L}}$ by the choice of $\mathcal{L}$ such that

$$C_{\mathrm{Lip}} - \varepsilon_{\mathcal{L}}^{-2}\nu < 0\,. \tag{10}$$

As with (8), on account of the chain rule we obtain for an arbitrary δ between 0 and 1:

$$\begin{aligned}\frac{d}{dt}\|u_1(t) - u_2(t)\|_H^2 \le -2\nu\frac{1-\delta}{\varepsilon_{\mathcal{L}}^2}\|u_1(t) - u_2(t)\|_H^2 + 2C_{Lip}\|u_1(t) - u_2(t)\|_H^2 \\ + C_{\mathcal{L},\delta} \max_{l_i\in\mathcal{L}} |l_i(u_1(t) - u_2(t))|\,.\end{aligned}$$

Thus, if we have knowledge of the asymptotic behavior of the term determined by finitely many linear functionals, $\max_{l_i\in\mathcal{L}} |l_i(u_1(t) - u_2(t))|$, say, if this term tends to zero, then we can conclude (by the Gronwall inequality) that the term $\|u_1(t) - u_2(t)\|_H$ tends to zero as well. This implies that the asymptotic behavior of Eq. (5) is determined by finitely many functionals (for instance, finitely many Fourier coefficients).

In this paper we will use the following result.

Theorem 2.1. *Assume that a random dynamical system φ has an absorbing forward invariant random set B in V such that $\sup_{x\in B(\omega)}\|x\|_V^2$ is bounded by a tempered random variable and $t\to\sup_{x\in B(\theta_t\omega)}\|x\|_V^2$ is locally integrable. Let $\mathcal{L}=\{l_j : j=1,\ldots,N\}$ be a set of linear continuous and linearly independent functionals on V. Suppose there exist constants $c_{\mathcal{L}}>0$ and $C_{\mathcal{L}}>0$ and a measurable function $l_{\mathcal{L}}(x_1,x_2,\omega)$ which maps $V\times V\times\Omega$ in $\mathbb{R}$ such that $l_{\mathcal{L}}(x_1,x_2,\omega)\geq -c_{\mathcal{L}}$ uniformly with respect to $(x_1,x_2,\omega)\in V\times V\times\Omega$ and for $x_1(\omega)$, $x_2(\omega)\in B(\omega)$ we have*

$$V(t,\omega)-V(s,\omega)\leq C_{\mathcal{L}}\cdot\int_s^t \mathcal{N}_{\mathcal{L}}(\tau,\omega)d\tau \\ +\int_s^t l_{\mathcal{L}}(\varphi(\tau,\omega,x_1),\varphi(\tau,\omega,x_2),\theta_\tau\omega)\cdot V(\tau,\omega)d\tau \tag{11}$$

for all $t\geq s\geq 0$, where

$$V(t,\omega)=\|\varphi(t,\omega,x_1)-\varphi(t,\omega,x_2)\|_H^2\,,$$

$$\mathcal{N}_{\mathcal{L}}(t,\omega)=\max_{j=1,\ldots,N}|l_j(\varphi(t,\omega,x_1)-\varphi(t,\omega,x_2))|^2\,.$$

Assume that

$$\frac{1}{t}\mathbb{E}\left\{\sup_{x_1,x_2\in B(\omega)}\int_0^t l_{\mathcal{L}}(\varphi(\tau,\omega,x_1),\varphi(\tau,\omega,x_2),\theta_\tau\omega)d\tau\right\}<0$$

for some $t>0$. Then $\mathcal{L}$ is a set of asymptotically determining functionals in probability for random dynamical system φ.

Proof. It is easy to find from relation (11) that

$$V(t,\omega)\leq V(0,\omega)e^{\int_0^t l_{\mathcal{L}}(s,\omega)ds}+C_{\mathcal{L}}\int_0^t \mathcal{N}_{\mathcal{L}}(\tau,\omega)e^{\int_\tau^t l_{\mathcal{L}}(s,\omega)ds}d\tau,$$

where $l_{\mathcal{L}}(t,\omega)=l_{\mathcal{L}}(\varphi(t,\omega,x_1),\varphi(t,\omega,x_2),\theta_t\omega)$. Therefore we can apply the argument given in the proof of Theorem 2.2 in [14]. □

Note that, in contrast to the above simple deterministic example (5), random partial differential equations which we derive from (1) do not have Lipschitz continuous nonlinearities. Therefore we have to consider the dynamics on a bounded set $B(\omega)$. Then $l_{\mathcal{L}}$ is related to the *local* Lipschitz constant on $B(\omega)$. In particular, $l_{\mathcal{L}}$ is related to the left-hand side of (10). Moreover, since the coefficients of random partial differential equations depend on ω, the set B and the constant $l_{\mathcal{L}}$ depend on ω as well.

We also note that the nonlinear structure of the two-layer fluid system does not allow us to reduce the situation to the direct application of the known results; see [8], [13], [14], [19]. Our main results use functionals supported by stream functions, whereas leading dynamical variables are the vorticities.

In Sec. 3, we return to the two-layer quasigeostrophic flow model.

3. Well-Posedness of the Random Two-Layer Fluid System

In the following, $L^2_{\rm per}$, $H^s_{\rm per}$ for $s \in \mathbb{R}$ are the standard Lebesgue and Sobolev spaces of L-periodic functions with the zero mean value, i.e. $\int_O \psi dO = 0$. Let $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ denote the standard scalar product and norm in $L^2_{\rm per}$, respectively. Every function $u(x,y)$ in $L^2_{\rm per}$ can be represented by the Fourier expansion

$$u(x,y) = \sum_{j\in\mathbb{Z}^2, j\neq 0} u_j L^{-1} \exp\left\{ i\frac{2\pi}{\mathcal{L}}(j_1 x + j_2 y)\right\},$$

where the Fourier coefficients u_j possess the property $\bar{u}_j = u_{-j}$ (bar denotes the complex conjugation) and

$$\|u\|_0^2 := \int_O |u|^2 dO \equiv \sum_{j\in\mathbb{Z}^2, j\neq 0} |u_j|^2 < \infty\,.$$

Note that $L^{-1}\exp\left\{i\frac{2\pi}{L}(j_1x + j_2y)\right\}$ is the eigenfunction of $-\Delta$ corresponding to the eigenvalue $\lambda_1 \cdot (j_1^2 + j_2^2)$, where $\lambda_1 = (2\pi/L)^2$ is the smallest positive eigenvalue. The norm in $H^s_{\rm per}$ is defined by the formula

$$\|u\|_s^2 := \int_O |(-\Delta)^{s/2} u|^2 dO \equiv \lambda_1^s \sum_{j\in\mathbb{Z}^2, j\neq 0} (j_1^2 + j_2^2)^s |u_j|^2\,.$$

It is clear that

$$\|\nabla u\|_s^2 := \|\partial_x u\|_s^2 + \|\partial_y u\|_s^2 = \|u\|_{1+s}^2\,, \quad s\in\mathbb{R}\,.$$

We also denote $\mathbf{L}^2_{\rm per} = L^2_{\rm per} \times L^2_{\rm per}$ and $\mathbf{H}^s_{\rm per} = H^s_{\rm per} \times H^s_{\rm per}$.

We work on the phase space $\mathbf{H}^{-1}_{\rm per}$ with the scalar product

$$(q,\bar{q})_* = h_1(\nabla\psi_1, \nabla\bar{\psi}_1)_0 + h_2(\nabla\psi_2, \nabla\bar{\psi}_2)_0 + p(\psi_1 - \psi_2, \bar{\psi}_1 - \bar{\psi}_2)_0\,,$$

where $q = (q_1, q_2)$, $\bar{q} = (\bar{q}_1, \bar{q}_2)$ and $\psi = (\psi_1, \psi_2) \in \mathbf{H}^1_{\rm per}$, $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$. The relation between q (resp. $\bar{q}$) and ψ (resp. $\bar{\psi}$) is defined by (2). Here we also use the notation

$$p = \frac{f_0^2}{g}\frac{\rho}{\rho_2 - \rho_1}\,, \quad \text{where} \quad F_1 h_1 = F_2 h_2 = p\,. \tag{12}$$

The norm induced by this scalar product

$$\|q\|_*^2 = (q,q)_* = h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2 + p\|\psi_1 - \psi_2\|_0^2$$

is equivalent to the usual norm on $\mathbf{H}_{\text{per}}^{-1}$. Moreover, we have the estimate

$$h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2 \le \|q\|_*^2 \le a_0\left(h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2\right), \tag{13}$$

where

$$a_0 = 1 + \frac{2p}{\lambda_1 \min\{h_1, h_2\}} = 1 + \frac{2}{\lambda_1}\max\{F_1, F_2\}. \tag{14}$$

To treat the nonlinearity in the two-layer fluid model we need the following lemma:

Lemma 3.1. *The Jacobian operator has the following properties*:

$$J(u,v) = -J(v,u), \quad (J(u,v),v)_0 = 0, \tag{15}$$

$$(J(u,v),w)_0 = (J(v,w),u)_0, \tag{16}$$

for u, v, w in H_{per}^1. Moreover the following estimates hold:

$$|(J(u,v),\Delta u)_0| \le c_0\|\Delta v\|_0 \cdot \|\nabla u\|_0 \cdot \|\Delta u\|_0, \quad u,\, v \in H_{\text{per}}^2; \tag{17}$$

$$\begin{aligned} |(J(u,v),w)_0| &\le c_1\|\Delta u\|_0 \cdot \|\Delta v\|_0 \cdot \|w\|_0, \quad u,\, v \in H_{\text{per}}^2,\, w \in L_{\text{per}}^2; \\ |(J(u,v),w)_0| &\le c_1\|\nabla u\|_0 \cdot \|\Delta v\|_0 \cdot \|\nabla w\|_0, \quad u,\, w \in H_{\text{per}}^1,\, v \in H_{\text{per}}^2, \end{aligned} \tag{18}$$

where $c_0 = (\sqrt{2} + (\sqrt{2}\cdot\pi)^{-1})^{1/2}$ and $c_1 = c_0\lambda_1^{-1/2}$.

Proof. Three identities may be easily verified. We look at the estimate (17). Note that

$$\begin{aligned} J(u,v)\cdot\Delta u &= \frac{1}{2}\cdot v_y\left\{\partial_x\left(u_x^2 - u_y^2\right) + 2\partial_y\left(u_x u_y\right)\right\} \\ &\quad + \frac{1}{2}\cdot v_x\left\{\partial_y\left(u_x^2 - u_y^2\right) - 2\partial_x\left(u_x u_y\right)\right\}. \end{aligned}$$

We then use the well-known Hölder inequality and the estimate (e.g., [20])

$$\|u\|_{L^4} \le c_0\|u\|_0^{1/2}\|\nabla u\|_0^{1/2},$$

to get the estimate (17).

In a similar way we can establish the other estimates in the lemma. □

Since the stochastic two-layer model (1) itself does not define a random dynamical system [1], we transform it into a system of coupled random partial differential equations, which will define a random dynamical system to which we can apply Theorem 2.1. The reason for taking such a transformation is that we need some particular *a priori* estimates for the trajectories of the system. Often these *a priori* estimates can be calculated by the Gronwall inequality. For equations containing white noise there exists no Gronwall inequality. But we can use this technique for the transformed random partial differential equations.

For this purpose we introduce an Ornstein–Uhlenbeck process $\eta(x, y, t, \omega)$ in $L^2_{\rm per}$. This process is defined by the solution of the following linear stochastic partial differential equation

$$\frac{\partial \eta}{\partial t} = \nu(k+1)\Delta\eta + \dot{W}\,, \tag{19}$$

with periodic boundary condition and with some initial condition $x \in L^2_{\rm per}$ where $k > 0$ is a free *control* parameter and W is a Wiener process in $L^2_{\rm per}$. We suppose that the covariance operator Q of this Wiener process has a finite trace. As mentioned in Sec. 2 such a Wiener process generates a metric dynamical system denoted by $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ where θ is the Wiener shift introduced in (3). It is well known that this equation has a stationary solution which is generated by a *Gaussian* random variable η in $H^1_{\rm per}$. In particular, the mapping

$$t \to \eta(\theta_t\omega) \in L^2_{\rm loc}(-\infty, \infty; H^1_{\rm per})$$

solves Eq. (19). Moreover, we can assume that the random variable η and the process $(t, \omega) \to \eta(\theta_t\omega)$ are defined for*all* $\omega \in \Omega$ what follows by a perfection argument for Ornstein–Uhlenbeck prcesses in Hilbert spaces, see Chueshov and Scheutzov [15], Proposition 3.1. For moments of η we obtain due to [34]:

$$\mathbb{E}\|\eta\|_1^2 \le \frac{{\rm tr}_0 Q}{2\nu(k+1)}\,, \quad \mathbb{E}\|\eta\|_1^{2n} \le C_n\left(\frac{{\rm tr}_0 Q}{\nu(k+1)}\right)^n, \; n \in \mathbb{N},\; C_n > 0\,. \tag{20}$$

We introduce new variables

$$\tilde{q}_1 := q_1 - \eta\,, \quad q_2,\; \tilde{\psi}_1 := \psi_1 + \xi_1\,, \quad \tilde{\psi}_2 := \psi_2 + \xi_2\,, \tag{21}$$

where the stationary process η solves the problem (19) and ξ_1 and ξ_2 are defined such that the elliptic equations (2) keep the same form

$$\tilde{q}_1 = \Delta\tilde{\psi}_1 - F_1 \cdot (\tilde{\psi}_1 - \tilde{\psi}_2)\,,$$

$$q_2 = \Delta\tilde{\psi}_2 - F_2 \cdot (\tilde{\psi}_2 - \tilde{\psi}_1)\,.$$

The processes ξ_1 and ξ_2 are solutions of the linear elliptic equations

$$\begin{aligned} &\Delta\xi_1 - F_1 \cdot (\xi_1 - \xi_2) = -\eta\,, \\ &\Delta\xi_2 - F_2 \cdot (\xi_2 - \xi_1) = 0\,, \end{aligned} \tag{22}$$

and can be presented in the form

$$\xi_1 = \frac{1}{F_1 + F_2}\{F_2(-\Delta)^{-1} + F_1(-\Delta + F_1 + F_2)^{-1}\}\eta\,,$$

$$\xi_2 = \frac{F_2}{F_1 + F_2}\left\{(-\Delta)^{-1} - (-\Delta + F_1 + F_2)^{-1}\right\}\eta\,,$$

where $(\cdot)^{-1}$ is the Green operator of the associated differential operator. Thus the processes ξ_1 and ξ_2 are smoother in spatial variables than η. In fact after simple calculations we have the estimates

$$\|\xi_i\|_{s+2} \le \|\eta\|_s\,, \quad i = 1, 2\,, \quad \text{and} \quad \|\xi_1 - \xi_2\|_{s+2} \le \|\eta\|_s \quad s \in \mathbb{R}\,.$$

Now we obtain the coupled random partial differential equations for new potential vorticities $\tilde{q}_1$, q_2:

$$\frac{\partial \tilde{q}_1}{\partial t} + J(\tilde{\psi}_1 - \xi_1, \tilde{q}_1 + \eta + \beta y) = \nu\Delta^2\tilde{\psi}_1 + f - \nu\Delta^2\xi_1 - \nu(k+1)\Delta\eta\,,$$

$$\frac{\partial q_2}{\partial t} + J(\tilde{\psi}_2 - \xi_2, q_2 + \beta y) = \nu\Delta^2\tilde{\psi}_2 - r\Delta\tilde{\psi}_2 - \nu\Delta^2\xi_2 + r\Delta\xi_2\,.$$

We treat η, ξ_1, ξ_2 as known processes. Using (22) we have

$$-\nu\Delta^2\xi_1 - \nu(k+1)\Delta\eta = -\nu - F_1(\Delta\xi_1 - \Delta\xi_2) - \nu k\Delta\eta,$$

$$-\nu\Delta^2\xi_2 = -\nu F_2(\Delta\xi_2 - \Delta\xi_1)\,.$$

For convenience, we drop the tilde and rewrite the above system. Thus we finally get the coupled system of random partial differential equations

$$\begin{aligned}\frac{\partial q_1}{\partial t} + J(\psi_1 - \xi_1, q_1 + \eta + \beta y) &= \nu\Delta^2\psi_1 + f \\ &\quad - \nu F_1(\Delta\xi_1 - \Delta\xi_2) - \nu k\Delta\eta, \\ \frac{\partial q_2}{\partial t} + J(\psi_2 - \xi_2, q_2 + \beta y) &= \nu\Delta^2\psi_2 - r\Delta\psi_2 \\ &\quad - \nu F_2(\Delta\xi_2 - \Delta\xi_1) + r\Delta\xi_2\,,\end{aligned} \tag{23}$$

with the coupling condition

$$\begin{aligned} q_1 &= \Delta\psi_1 - F_1\cdot(\psi_1 - \psi_2)\,, \\ q_2 &= \Delta\psi_2 - F_2\cdot(\psi_2 - \psi_1)\,, \end{aligned} \tag{24}$$

in the class of L-periodic functions with initial data $q(x,y,0) = q_0(x,y) \equiv (q_{01}(x,y), q_{02}(x,y)) \in \mathbf{H}^{-1}_{\mathrm{per}}$, where η is the stationary solution to (19) and ξ_1 and ξ_2 solve (22) in H^2_{per}.

For the rest of the paper, we work on this coupled system (23) of random partial differential equations for the stochastically forced two-layer quasigeostrophic fluid system.

The coefficients of coupled system (23), (24) have properties similar to those of the coefficients of the corresponding deterministic two-layer quasigeostrophic system (cf. [6], where an N-layer model with other boundary conditions is considered). Therefore, as in [6], using the Galerkin method and the compactness argument we can prove the following assertion on the well-posedness of problem (23), (24) in the class of L-periodic functions.

Theorem 3.1. *Let $q_0 \in \mathbf{H}^{-1}_{\mathrm{per}}$ and $f \in L^2_{\mathrm{per}}$. Then for all $\omega \in \Omega$ and for all $\tau > 0$, the system* (23), (24) *has a unique solution $\{q(t), \psi(t)\}$ such that*

$$q \in C([0,\infty); \mathbf{H}^{-1}_{\mathrm{per}}) \cap L^2_{\mathrm{loc}}(0,\infty; \mathbf{L}^2_{\mathrm{per}}) L^2_{\mathrm{loc}}(\tau,\infty; \mathbf{H}^1_{\mathrm{per}}).$$

The function ψ *associated to* q *by* (24) *satisfies*

$$\psi \in C([0,\infty);\mathbf{H}^1_{\mathrm{per}})L^2_{\mathrm{loc}}(0,\infty;\mathbf{H}^2_{\mathrm{per}})L^2_{\mathrm{loc}}(\tau,\infty;\mathbf{H}^3_{\mathrm{per}}).$$

The solution depends continuously on the initial condition $q_0 \in \mathbf{H}^{-1}_{\mathrm{per}}$.

By the uniqueness assertion of the last theorem the solution $t \to q(t)$ generates a random dynamical system φ. Moreover, the mapping $\varphi:\ \mathbf{H}^{-1}_{\mathrm{per}} \to \mathbf{H}^{-1}_{\mathrm{per}}$ which maps q_0 to $\varphi(t,\omega,q_0)$ is continuous.

4. Dissipativity of the Random Two-Layer Fluid System

Dynamical systems generated by many nonlinear parabolic differential equations have the dissipative property which means that there exists a set absorbing the states of the system in finite time. Usually one can choose such a set which is also compact and forward invariant.

We now construct an absorbing forward invariant set for the random dynamical system generated by (23). This set will be a random set.

Theorem 4.1. *There exists a compact random set* $B(\omega) \subset \mathbf{H}^{-1}_{\mathrm{per}}$ *such that*

$$\begin{aligned} \varphi(t,\omega,B(\omega)) &\subset B(\theta_t\omega) \quad \mathit{for} \quad t \geq 0\,,\\ \varphi(t,\omega,q(\omega)) &\subset B(\theta_t\omega) \quad \mathit{for} \quad t \geq t_0(\omega,q)\,, \end{aligned} \tag{25}$$

where q *is a random variable with values in* $\mathbf{H}^{-1}_{\mathrm{per}}$.

We now divide the proof of this theorem into some lemmas. We start with the following lemma.

Lemma 4.1. *Let* $q(t)$ *be the solution of* (23). *Then* $q(t)$ *satisfies the following inequality*

$$\begin{aligned} &\frac{d}{dt}\|q(t)\|_*^2 + \nu(h_1\|\Delta\psi_1(t)\|_0^2 + h_2\|\Delta\psi_2(t)\|_0^2)\\ &\qquad \leq d_0\cdot\|\eta(\theta_t\omega)\|_0^2\cdot(h_1\|\nabla\psi_1\|_0^2 + h_2\|\nabla\psi_2\|_0^2) + m(\theta_t\omega)\,, \end{aligned}$$

where

$$m(\omega) = d_1\|\eta(\omega)\|_0^4 + d_2\|\eta(\omega)\|_0^2 + d_3$$

and

$$\begin{aligned} d_0 &= \frac{6c_0^2}{\nu}\left(1 + \frac{p^2\nu}{\lambda_1^2\min\{h_1,h_2\}}\right),\\ d_1 &= \frac{6c_0^2h_1}{\nu\lambda_1}\,,\\ d_2 &= 9\left(\frac{\beta^2(h_1+h_2)}{\nu\lambda_1^3} + \frac{\nu p^2}{\lambda_1^2}\left(\frac{1}{h_1} + \frac{1}{5h_2}\right) + \frac{rh_2}{18\lambda_1} + k^2\nu h_1\right),\\ d_3 &= \frac{9h_1}{\nu\lambda_1}\|f\|_{-1}^2\,. \end{aligned}$$

We omit the proof of this technical but simple lemma. The proof is based on the chain rule applied to $\|q\|_*^2$ and properties of J; see Lemma 3.1.

We now consider the random variable η defined in Sec. 2. Recall that η depends on the control parameter k. If k is chosen large enough, then particular moments of η are small. Especially we can formulate:

Lemma 4.2. *Let W be a Wiener process in L^2_{per} with finite trace of the covariance. Then under assumptions*

$$\frac{2d_0 a_0 \mathrm{tr}_0 Q}{\lambda_1^2 \nu^2 (k+1)} < 1\,, \quad \frac{16 d_0 \mathrm{tr}_0 Q}{\lambda_1^2 \nu^2 (k+1)^2} < 1 \tag{26}$$

the random variable

$$R_0(\omega) := \int_{-\infty}^0 e^{\frac{\nu\lambda_1}{a_0}\tau + d_0 \int_\tau^0 \|\eta(\theta_{\tau'}\omega)\|_0^2 d\tau'} m(\theta_\tau\omega) d\tau$$

is finite and tempered. Moreover

$$(\mathbb{E}R_0^2)^{1/2} \le d_4^2 \left(\frac{3a_0}{2\nu\lambda_1}\right)^{3/2} \cdot \left(\frac{2\lambda_1\nu}{a_0} - \frac{4d_0 \mathrm{tr}_0 Q}{\lambda_1 \nu (k+1)}\right)^{-1/2},$$

where

$$d_4 = C_8^{1/4} \frac{d_1 (\mathrm{tr}_0 Q)^2}{\lambda_1^2 \nu^2 (k+1)^2} + C_4^{1/4} \frac{d_2 \mathrm{tr}_0 Q}{\lambda_1 \nu (k+1)} + d_3$$

is an estimate for $(\mathbb{E}m^4)^{1/4}$ (*the constants* C_8, C_4 *are defined in* (20)).

The proof of this lemma can be found in Chueshov *et al.* [14] for an Ornstein–Uhlenbeck process in another Hilbert space. However the argument given there is of a general nature.

We now construct a set satisfying (25).

Lemma 4.3. *Let $R(\omega) := aR_0(\omega)$ for some $a > 1$ and R_0 as in Lemma* 4.2. *Then the closed $\mathbf{H}^{-1}_{\mathrm{per}}$-ball $B(0, R(\omega)^{1/2})$ fulfills* (25) *provided conditions* (26) *hold.*

Proof. Using Lemma 4.1 and relation (13) we have

$$\frac{d}{dt}\|q(t)\|_*^2 \le \left(-\frac{\nu\lambda_1}{a_0} + d_0 \cdot \|\eta(\theta_t\omega)\|_0^2\right) \cdot \|q(t)\|_*^2 + m(\theta_t\omega).$$

Let $q_0 = q(0)$ and $\rho(t, \omega, \|q_0\|_*^2)$ be the solution of the one-dimensional random affine equation

$$\frac{d\rho(t)}{dt} + \frac{\nu\lambda_1}{a_0}\rho = d_0\|\eta(\theta_t\omega)\|_0^2 \rho + m(\theta_t\omega)\,, \quad \rho(0, \omega, \|q_0\|_*^2) = \|q_0\|_*^2\,. \tag{27}$$

A comparison argument gives that

$$\|\varphi(t, \omega, q_0)\|_*^2 \equiv \|q(t)\|_*^2 \le \rho(t, \omega, \|q_0\|_*^2)\,.$$

Here φ is the dynamical system introduced in Sec. 3: $\varphi(t, \omega, q_0) = q(t)$, where $q(t)$ is the solution to (23) with the initial data q_0. Equation (27) has the stationary solution given by $t \to R_0(\theta_t \omega)$:

$$\rho(t, \omega, R_0(\omega)) = R_0(\theta_t \omega)\,.$$

This can be checked by the variation of constants formula. This solution is exponentially attracting, as seen from the variation of constants formula again:

$$\begin{aligned}|R_0(\theta_t\omega) - \rho(t, \omega, \|q_0\|_*^2)| &= |\rho(t, \omega, R_0(\omega)) - \rho(t, \omega, \|q_0\|_*^2)| \\ &\le e^{\int_0^t \left(d_0 \|\eta(\theta_\tau \omega)\|_0^2 - \frac{\nu\lambda_1}{a_0}\right) d\tau} \left(R(\omega) + \|q_0\|_*^2\right)\end{aligned}$$

which tends to zero exponentially. Indeed, it follows from (20) that for a sufficient small $\varepsilon > 0$,

$$\int_0^t d_0 \|\eta(\theta_\tau \omega)\|_0^2 d\tau < \frac{\nu\lambda_1 - \varepsilon}{a_0} t\,, \qquad \frac{\nu\lambda_1 - \varepsilon}{a_0} > 0$$

for large $t > 0$. □

It remains to prove the existence of a compact set B satisfying (25).

Lemma 4.4. *Suppose that the random variable $R(\omega)$ is defined in Lemma* 4.3. *The set*

$$B(\omega) := \overline{\varphi(1, \theta_{-1}\omega, B(0, R(\theta_{-1}\omega)^{1/2}))}$$

is a compact absorbing forward invariant random set in $\mathbf{H}^{-1}_{\text{per}}$. Moreover

$$\omega \mapsto \sup\{\|\Delta\psi_1\|_0^2 + \|\Delta\psi_2\|_0^2 \;:\; (q_1, q_2) \in B(\omega)\} \tag{28}$$

is a tempered random variable (ψ_1 and ψ_2 are defined by (24)).

Proof. The regularity assertion of Theorem 3.1 and some standard techniques (see Bernier [6]) imply that the sets $B(\omega) \subset \mathbf{H}^{-1}_{\text{per}}$ are compact. Since R is a random variable, the ball $B(0, R^{1/2})$ is a random set. The continuity of $\varphi(t, \omega, \cdot)$ allows us to conclude that B is a random set. The construction of B ensures that the set is absorbing and forward invariant. The temperedness of (28) can be proved in the same way as in [14] for the 2D Navier–Stokes equations. □

Putting the above four lemmas together, we obtain Theorem 4.1.

In the next section we consider asymptotic probabilistic determining functionals for the random two-layer fluid system.

5. Determining Functionals for the Random Two-Layer Fluid System

In this section we investigate, under what conditions, the stochastically forced two-layer quasigeostrophic fluid system (1) or (23) has a set of asymptotically determining functionals in probability. Namely, we are going to derive conditions such that the two-layer fluid problem satisfies the assumptions of Theorem 2.1. In particular, we will estimate the constant $l_{\mathcal{L}}$. To this end, we consider the difference of two solutions $\hat{q}$ and $\bar{q}$ of (23), corresponding to the stream functions $\hat{\psi}$, $\bar{\psi}$. We set

$$q = (q_1, q_2) = \hat{q} - \bar{q},$$

with the associated stream function

$$\psi = \hat{\psi} - \bar{\psi}.$$

We derive the random partial differential equations for the solution difference $q = \hat{q} - \bar{q}$ from (23):

$$\frac{\partial q_1}{\partial t} = \nu\Delta^2\psi_1 - J(\psi_1, \hat{q}_1 + \beta y) - J(\bar{\psi}_1, q_1) - J(\psi_1, \eta) + J(\xi_1, q_1)\,, \tag{29}$$

$$\frac{\partial q_2}{\partial t} = \nu\Delta^2\psi_2 - r\Delta\psi_2 - J(\psi_2, \hat{q}_2 + \beta y) - J(\bar{\psi}_2, q_2) + J(\xi_2, q_2)\,, \tag{30}$$

Multiplying (29) by $-h_1\psi_1$, and (30) by $-h_2\psi_2$ and adding together, we have the equality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q\|_*^2 = &-\nu[h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2] - rh_2\|\nabla\psi_2\|_0^2\\ &+ h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 + h_2(J(\bar{\psi}_2 - \xi_2, q_2), \psi_2)_0\,.\end{aligned} \tag{31}$$

To estimate the nonlinear term containing J we obtain the following result.

Lemma 5.1. *We have for* ξ_1, $\xi_2 \in \mathbf{H}^2_{\mathrm{per}}$

$$\begin{aligned}&2h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 + 2h_2(J(\bar{\psi}_2 - \xi_2, q_2), \psi_2)_0\\ &\quad\le b_0(h_1\|\bar{\psi}_1 - \xi_1\|_2^2\|\nabla\psi_1\|_0^2 + h_2\|\bar{\psi}_2 - \xi_2\|_2^2\|\nabla\psi_2\|_0^2)\\ &\qquad+ \nu(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2)\,,\end{aligned}$$

where

$$b_0 = \frac{2c_0^2}{\nu}(1 + F_1F_2\lambda_1^{-2}) \tag{32}$$

and F_1, F_2, c_0 *are defined in* (2) *and Lemma* 3.1.

Proof. We decompose the nonlinearity as

$$h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 \equiv I_1 + I_2,$$

where the two parts are defined by

$$I_1 = h_1(J(\bar{\psi}_1 - \xi_1, \Delta\psi_1), \psi_1)_0\,, \quad I_2 = p(J(\bar{\psi}_1 - \xi_1, \psi_2), \psi_1)_0\,,$$

see (24). By (17) we can deduce

$$|I_1| \le c_0 h_1 \|\bar{\psi}_1 - \xi_1\|_2 \|\Delta\psi_1\|_0 \|\nabla\psi_1\|_0\,.$$

Using (18) we obtain

$$|I_2| \le c_1 p \|\bar{\psi}_1 - \xi_1\|_2 \|\nabla\psi_2\|_0 \|\nabla\psi_1\|_0 \le c_1 p \lambda_1^{-1/2} \|\bar{\psi}_1 - \xi_1\|_2 \|\Delta\psi_2\|_0 \|\nabla\psi_1\|_0\,.$$

According to the inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ and (12), we can estimate

$$\begin{aligned} h_1(J(\bar{\psi}_1 - \xi_1, q_1), \psi_1)_0 \le{}& \frac{\nu}{4}(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2) \\ &+ \frac{h_1 c_0^2}{\nu}(1 + F_1F_2\lambda_1^{-2})\|\bar{\psi}_1 - \xi_1\|_2^2 \|\nabla\psi_1\|_0^2\,. \end{aligned}$$

In a similar way we have for $c_1 = \lambda_1^{-1/2} c_0$ (see Lemma 3.1)

$$\begin{aligned} h_2(J(\bar{\psi}_2 - \xi_2, q_2), \psi_2)_0 \le{}& \frac{\nu}{4}(h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2) \\ &+ \frac{h_2 c_0^2}{\nu}(1 + F_1F_2\lambda_1^{-2})\|\bar{\psi}_2 - \xi_2\|_2^2 \|\nabla\psi_2\|_0^2\,. \qquad \square \end{aligned}$$

On account of (31) and the last lemma we obtain

$$\begin{aligned} \frac{d}{dt}\|q\|_*^2 \le{}& -\nu[h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2] - 2rh_2\|\nabla\psi_2\|_0^2 \\ &+ b_0(h_1\|\bar{\psi}_1 - \xi_1\|_2^2\|\nabla\psi_1\|_0^2 + h_2\|\bar{\psi}_2 - \xi_2\|_2^2\|\nabla\psi_2\|_0^2)\,. \end{aligned} \tag{33}$$

This estimate is the main point in the construction of determining functionals. It is our aim to obtain an integral equation like (11). To this end we first consider the case when Ekman parameter $r > 0$ is arbitrary.

Let $\mathcal{L} = \{l_j\}_1^N$ be a set of linearly independent bounded linear functionals on the space $\mathbf{H}^2_{\mathrm{per}}$. Assume that the set $\mathcal{L}$ possesses the property

$$\|\psi\|_1 \le C_{\mathcal{L}} \max_j |l_j(\psi)| + \varepsilon_{\mathcal{L}}\|\psi\|_2\,, \tag{34}$$

for any $\psi \in \mathbf{H}^2_{\mathrm{per}}$ with some positive constant $C_{\mathcal{L}}$ and $\varepsilon_{\mathcal{L}}$ (compare with (9)). We note that the best possible value of the parameter $\varepsilon_{\mathcal{L}}$ is called the *completeness defect* of the family $\mathcal{L}$ with respect to the pair of spaces $\mathbf{H}^1_{\mathrm{per}}$ and $\mathbf{H}^2_{\mathrm{per}}$ (see [12], [11]). We refer to [12], [11] for general properties of the completeness defect and for estimates of ε_L for several families of functionals on Sobolev spaces. It is crucial for the following considerations to choose a set $\mathcal{L}$ such that $\varepsilon_{\mathcal{L}}$ is *small*. From (34) for $\psi = (\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)$ we have

$$\begin{aligned} (h_1\|\Delta\psi_1\|_0^2 + h_2\|\Delta\psi_2\|_0^2) \ge{}& \frac{1-\delta}{\varepsilon_L^2}(h_1\|\nabla\psi_1\|_0^2 + \|h_2\nabla\psi_2\|_0^2) \\ &- C_{\delta,\mathcal{L}} \max_{j=1,\dots,N} |l_j(\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)|^2 \end{aligned}$$

for any $0 < \delta < 1$ with appropriate positive constant $C_{\delta,\mathcal{L}}$. Therefore from (33) and (13) we obtain

$$\frac{d}{dt}\|q\|_*^2 \leq l_{\mathcal{L}}(\bar{\psi},\omega)\|q\|_*^2 + C_{\delta,\mathcal{L}} \max_{j=1,\ldots,N} |l_j(\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2)|^2 \,, \tag{35}$$

where

$$l_{\mathcal{L}}(\bar{\psi},\omega) = -\frac{1-\delta}{a_0\varepsilon_{\mathcal{L}}^2} + b_0 \max\left\{\|\bar{\psi}_1 - \xi_1\|_2^2 \,, \|\bar{\psi}_2 - \xi_2\|_2^2\right\}$$

and a_0, b_0 are defined in (14), (32), respectively. By Theorem 2.1 we obtain the following assertion.

Theorem 5.1. *Let $\mathcal{L} = \{l_j : j = 1,\ldots,N\}$ be a finite set of linearly independent continuous functionals on the space $\mathbf{H}^2_{\mathrm{per}}$. We assume that this set satisfies* (34). *Let* (26) *hold. If*

$$\varepsilon_{\mathcal{L}} < \frac{\nu\sqrt{\min\{h_1,h_2\}}}{\sqrt{2a_0b_0\Sigma}},$$

where

$$\Sigma = d_0(\mathbb{E}\|\eta\|_0^4)^{1/2}(\mathbb{E}R^2)^{1/2} + \mathbb{E}m + \nu\min\{h_1,h_2\}\mathbb{E}\|\eta\|_0^2 \tag{36}$$

and m, involving the Wiener process through the Ornstein–Uhlenbeck process η, is defined in Lemma 4.1, *then $\mathcal{L}^{h_1,h_2} = \{l_j^{h_1,h_2} : j = 1,\ldots,N\}$ where $l_j^{h_1,h_2}(\psi_1,\psi_2) := l_j(\sqrt{h_1}\psi_1, \sqrt{h_2}\psi_2), l_j \in \mathcal{L}$ is a set of asymptotically determining functionals in probability for the stochastically forced two-layer quasigeostrophic fluid system* (23), (24).

Proof. In order to check the conditions of Theorem 2.1 we integrate (35). $l_{\mathcal{L}}$ can be estimated by

$$-\nu\frac{1-\delta}{a_0\varepsilon_{\mathcal{L}}^2} + \frac{2\nu b_0}{\nu\min\{h_1,h_2\}}(h_1\|\Delta\bar{\psi}_1\|_0^2 + h_2\|\Delta\bar{\psi}_2\|_0^2) + 2b_0\|\eta\|_0^2 \,.$$

On account of Lemma 4.1,

$$\sup_{x\in B(\omega)} \frac{1}{t}\nu\int_0^t (h_1\|\Delta\bar{\psi}_1(\tau,\omega,x)\|_0^2 + h_2\|\Delta\bar{\psi}_2(\tau,\omega,x)\|_0^2)d\tau$$

has the bound

$$\frac{1}{t}\left(R(\omega) + \int_0^t (d_0\|\eta(\theta_\tau\omega)\|_0^2 R(\theta_\tau\omega) + m(\theta_\tau\omega))d\tau\right)$$

since B is forward invariant. Note that the expectation of R/t can be made arbitrarily small if t is large. The conclusion follows by calculating the expectation of the last expression and choosing t sufficiently large. $\square$

Remark 5.1. (i) Note that the main task is to prove the existence of determining functionals of Eq. (1). It is easily seen, due to the structure of the transformations (21), that a set $\mathcal{L}$ determines in probability for (23) if and only if the same set $\mathcal{L}$

is determining in probability for (1). Indeed, the transformation (21) between the solution of (1) and (23) is linear.

(ii) It follows straightforwardly that there exists a set $\hat{\mathcal{L}}^{h_1,h_2}$ of linearly independent linear bounded functionals $\{\hat{l}_j\}_1^N$ which are determining with respect to q. Indeed, q and ψ are connected by a linear homeomorphism Λ from $\mathbf{L}^2_{\rm per}$ to $\mathbf{H}^2_{\rm per}$ defined by (24) such that we can set $\hat{l}_j^{h_1,h_2} = l_j^{h_1,h_2} \circ \Lambda$ where $\mathcal{L} = \{l_j\}_1^N$ defines the set of determining functionals introduced in Theorem 5.1.

(iii) Assumption (26) holds if

$$\frac{4d_0 a_0 {\rm tr}_0 Q}{\lambda_1^2 \nu^2 (k+1)} < 1\,, \quad k+1 > \frac{4}{a_0}\,, \tag{37}$$

for example. In this case it is easy to see that $\left(\mathbb{E}R_0^2\right)^{1/2} \le 4d_4^2 \left(\frac{3a_0}{2\nu\lambda_1}\right)^2$ (we choose $a = 4/3$ in Lemma 4.3). Therefore using (20) we obtain

$$\Sigma \le C_4^{1/2} \frac{{\rm tr}_0 Q}{\nu(k+1)} \cdot \left(4d_0 d_4^2 \left(\frac{3a_0}{2\nu\lambda_1}\right)^2 + \nu \min\{h_1, h_2\}\right) + d_4\,.$$

On account of (37) again we have the estimate

$$d_4 \le \frac{{\rm tr}_0 Q}{k+1} \cdot \left(C_8^{1/4} \frac{d_1}{4d_0 a_0} + C_4^{1/4} \frac{d_2}{\lambda_1 \nu}\right) + d_3\,.$$

Therefore under conditions (37) for small ${\rm tr}_0 Q$ we obtain estimate

$$\varepsilon_{\mathcal{L}} < \frac{\nu^2 \lambda_1 \sqrt{\min\{h_1, h_2\}}}{6\|f\|_{-1} c_0 \sqrt{h_1}} \cdot \left\{(\lambda_1 + 2\max\{F_1, F_2\})\left(1 + \frac{F_1 F_2}{\lambda_1^2}\right)\right\}^{-1/2}. \tag{38}$$

If this estimate holds, then the functionals $\{l_j^{h_1,h_2}\}$ are determining for the *deterministic* two-layer model. Moreover under condition (38) these functionals are also determining for our stochastic two-layer model (1) and (2) provided the noise parameter ${\rm tr}_0 Q$ is small enough.

(iv) Heuristically speaking, as seen in (38) for fixed parameters h_1, h_2 and decreasing $\nu > 0$, we know that $\varepsilon_{\mathcal{L}}$ must be decreasing. We further see that the number of functionals such that (34) is satisfied is increasing. Therefore the degree of freedom of the two-layer system is increasing if the viscosity ν is decreasing.

We will present two examples of determining functionals in Sec. 7.

6. Slaving Properties of the Random Two-Layer Fluid System

In this section, we further show that the long-time dynamics of the two-layer geophysical fluid system is determined by the long time dynamics of the top layer alone, when the fluid parameters and the Wiener process satisfy certain conditions. Specifically, we show that the long time dynamics is determined by finitely many linearly independent functionals defined only on the top layer.

Before we continue with the calculations, let us explain heuristically the physical reason for this special behavior. We suppose that the Ekman constant $r > 0$ is sufficiently large that it provides a large amount of friction or dissipation for the bottom layer. We introduce determining functionals for the top layer. The bottom layer has behavior similar to that of the system given by the simple deterministic equation (5), as long as condition (7) is satisfied. Note that the Ekman friction term or the value r contributes a negative part to the constant $l_{\mathcal{L}}$; see (33) or later on in (40). Thus when $r > 0$ is sufficiently large, the constant $l_{\mathcal{L}}$ becomes sufficiently negative. Thus larger r contributes to larger dissipation in the bottom layer, and it makes the bottom layer asymptotically slaved by the top layer.

Let $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ be a finite set of linearly independent continuous functionals on the space H^2_{per} of the top stream functions. We assume that

$$\|\nabla\psi_1\|_0 \le \varepsilon_{\mathcal{L}} \cdot \|\Delta\psi_1\|_0 + C_{\mathcal{L}} \cdot \max_{j=1,\dots,N} |l_j(\psi_1)|\,, \quad \psi_1 \in H^2_{\text{per}}\,, \tag{39}$$

where $C_{\mathcal{L}} > 0$ is a constant and $\varepsilon_{\mathcal{L}}$ is the completeness defect depending on $\mathcal{L}$. As above we have

$$h_1\|\Delta\psi_1\|_0^2 \ge \frac{1-\delta}{\varepsilon^2_{\mathcal{L}}} h_1\|\nabla\psi_1\|_0^2 - C_{\delta,\mathcal{L}} h_1 \max_{j=1,\dots,N} |l_j(\psi_1)|^2\,.$$

for any $0 < \delta < 1$ with appropriate positive constant $C_{\delta,\mathcal{L}}$.

Thus under condition (39) and using the inequality $\|\nabla\psi_2\|_0^2 \le \lambda_1\|\Delta\psi_2\|_0^2$, the estimate (33) for the solution difference $\hat{q} - \bar{q}$ becomes

$$\begin{aligned} &\frac{d}{dt}\|q\|_*^2 + \nu h_1 \frac{1-\delta}{\varepsilon^2_{\mathcal{L}}}\|\nabla\psi_1\|_0^2 + (\nu\lambda_1 + 2r) h_2 \|\nabla\psi_2\|_0^2 \\ &\qquad \le b_0 \max\left\{\|\bar{\psi}_1 - \xi_1\|_2^2\,, \|\bar{\psi}_2 - \xi_2\|_2^2\right\}\|q\|_*^2 + C_{\delta,\mathcal{L}} \max_{j=1,\dots,N} |l_j(h_1\psi_1)|^2\,. \end{aligned}$$

Consequently

$$\frac{d}{dt}\|q\|_*^2 \le l_{\mathcal{L}}(\bar{\psi}, \omega)\|q\|_*^2 + C_{\delta,\mathcal{L}} \max_{j=1,\dots,N} |l_j(h_1\psi_1)|^2, \tag{40}$$

where

$$l_{\mathcal{L}}(\bar{\psi}, \omega) = -\min\left\{\nu\frac{1-\delta}{a_0\varepsilon^2_{\mathcal{L}}}\,, \frac{\nu\lambda_1 + 2r}{a_0}\right\} + b_0 \max\left\{\|\bar{\psi}_1 - \xi_1\|_2^2\,, \|\bar{\psi}_2 - \xi_2\|_2^2\right\}\,.$$

Again applying Theorem 2.1 we obtain the main result.

Theorem 6.1. *Let $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ be a finite set of linearly independent continuous functionals on the space H^2_{per}. We assume that this set satisfies the following condition involving only the top fluid layer dynamical variable ψ_1:*

$$\|\nabla\psi_1\|_0 \le \varepsilon_{\mathcal{L}} \cdot \|\Delta\psi_1\|_0 + C_{\mathcal{L}} \cdot \max_{j=1,\dots,N} |l_j(\psi_1)|\,, \quad \psi_1 \in H^2_{\text{per}}\,. \tag{41}$$

If

$$\Sigma < \min\left\{\frac{\nu}{\varepsilon^2_{\mathcal{L}}}\,, (\nu\lambda_1 + 2r)\right\}\frac{\nu\min\{h_1, h_2\}}{2a_0b_0}\,, \tag{42}$$

where Σ is given by (36), then $\mathcal{L}$ is a set of asymptotically determining functionals in probability for the stochastically forced two-layer quasigeostrophic fluid system

(23), (24). *Therefore, the asymptotic probabilistic dynamics of the stochastically forced two-layer quasigeostrophic fluid system is determined only by the top fluid layer.*

The proof of this theorem is very similar to the proof of Theorem 5.1.

Relation (42) holds if

$$\frac{\nu}{\varepsilon_{\mathcal{L}}^2} \geq \nu\lambda_1 + 2r \tag{43}$$

and

$$\Sigma < (\nu\lambda_1 + 2r)\frac{\nu \min\{h_1, h_2\}}{2b_0 a_0}\,. \tag{44}$$

The parameter Σ depends on r via d_2. However as in Remark 5.1 for small enough $\mathrm{tr}_0 Q$ the estimate (44) turns into the relation

$$\nu\lambda_1 + 2r > \frac{36c_0^2\|f\|_{-1}^2 h_1}{\nu^3\lambda_1 \min\{h_1, h_2\}} \cdot \left(1 + \frac{2}{\lambda_1}\max\{F_1, F_2\}\right)\left(1 + \frac{F_1F_2}{\lambda_1^2}\right). \tag{45}$$

This relation requires that the Ekman constant r is sufficiently large. This observation leads to the following assertion.

Corollary 6.1. *Assume that* (45) *holds. Let* $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ *be a finite set of linearly independent continuous functionals on the space* H^2_{per}. *We assume that this set satisfies* (41) *with the parameter* $\varepsilon_{\mathcal{L}}$ *given in* (43). *Then* $\mathcal{L}$ *is a set of asymptotically determining functionals in probability for the stochastically forced two-layer quasigeostrophic fluid system* (23), (24) *provided* $\mathrm{tr}_0 Q$ *is small enough.*

Consequently, the asymptotic probabilistic dynamics of the stochastically forced two-layer quasigeostrophic fluid system is determined only by the top fluid layer.

As Theorem 5.1 and Theorem 6.1 show, the problem of the investigation of finite families of determining functionals is reduced to the study of sets of functionals for which the estimate (34) (or (39)) holds with $\varepsilon_{\mathcal{L}}$ small enough. It is also important to calculate the best possible value for $\varepsilon_{\mathcal{L}}$ for the given family of functionals. For this parameter there is the estimate from below depending only on a number of functionals [12], and this estimate coincides with $\varepsilon_{\mathcal{L}}$ when functionals are modes (see Example 7.1 below). We refer to [11], [12] for further discussions concerning an optimal choice of families of functionals with properties like (34) or (39). We also note that the deterministic counterparts of Theorems 5.1 and 6.1 were proved in [7] for other boundary conditions.

7. Examples and Applications

We now present two examples of asymptotically determining functionals in probability, and then discuss the conditions in our main results, Theorems 5.1 and 6.1, in the context of realistic geophysical parameters.

Example 7.1. (Determining modes) Let $\{e_i(x,y)\}_{i=1}^{\infty}$ be the basis of eigenfunctions of $-\Delta$ with the periodic boundary conditions in O such that the corresponding eigenvalues possesses the property

$$0 < \lambda_1 \leq \lambda_2 \leq \dots . \tag{46}$$

We note that every eigenfunction has the form $L^{-1}\exp\{\frac{2\pi}{L}(j_1x + j_2y)\}$. However we numerate them such that (46) holds. Suppose $\mathcal{L}_N = \{l_j : j = 1,\dots,N\}$ is the set of the functionals on $H^2_{\rm per}$ of the form

$$l_j(u) = \int_O u(x,y)e_j(x,y)dO\,, \quad j = 1,\dots,N\,.$$

Then one can prove (see, e.g. [12]) that estimate (39) holds for $\mathcal{L}_N$ with $\varepsilon_{\mathcal{L}_N} = \lambda_{N+1}^{-1/2}$ and this value is the best possible among all families of functionals consisting of N elements. Since $\lambda_N \sim c_0NL^{-2}$ for N large enough, we have that $\varepsilon_{\mathcal{L}_N} \sim \tilde{c}_0L/\sqrt{N}$ with some absolute constant $\tilde{c}_0$. Therefore we can use Theorem 6.1 to estimate the number of determining modes. We also note that using the family $\mathcal{L}_N$ of the functionals on $H^2_{\rm per}$ we can easily construct a family $\tilde{\mathcal{L}}_N$ of functionals on $\mathbf{H}^2_{\rm per} = H^2_{\rm per} \times H^2_{\rm per}$ such that (34) holds with $\varepsilon_{\mathcal{L}_N} = \lambda_{N+1}^{-1/2}$.

Example 7.2. (Determining nodes) Let us consider the nodes

$$(x_i, y_j) = \frac{L}{\sqrt{N}} \cdot (i,j) \in O\,, \quad 1 \leq i,j \leq \sqrt{N}\,,$$

and define functionals l_{ij} on $H^2_{\rm per}$ as δ-functions at (x_i, y_j), i.e by the formulas $l_{ij}(u) = u(x_i, y_j)$. Let $\mathcal{L} = \{l_{ij}\}$. One can prove (see, e.g. [12] or [27]) that estimate (39) holds for $\mathcal{L}$ with $\varepsilon_{\mathcal{L}} = \bar{c}_0L/\sqrt{N}$ where $\bar{c}_0 = 1/2$ is an absolute constant [27]. Thus $\varepsilon_{\mathcal{L}}$ has the same order for large N as the best possible value $\varepsilon_{\mathcal{L}_N}$ for families of functionals consisting of N elements.

We also note that the estimates for the completeness defect $\varepsilon_{\mathcal{L}}$ and the Ekman constant r given in Theorems 5.1 and 6.1 are rather crude. In the two-layer quasi-geostrophic model considered above, the parameter ν is the molecular viscosity. For ocean water, $\nu = 10^{-6}$ m^{2}s^{-1}. However, fluid turbulence at small scales can act as an extra dissipative mechanism, thus calling for the substitution of the molecular viscosity by a much larger eddy viscosity. For example, in the two-layer quasi-geostrophic model flow simulation in [31], the eddy viscosity (we still use the same notation as the molecular viscosity) is taken as $\nu = 50$ m^{2}s^{-1}. The gravitational acceleration is $g = 9.81$ ms^{-2}.

At mid-latitude (45°N), $f_0 = 8 \times 10^{-5}$ s^{-1} and $\beta = 2.3 \times 10^{-11}$ m^{-1}s^{-1}. For large-scale flows at mid-latitude, such as the Gulf stream in the Atlantic ocean, the horizontal spatial scale L is of the order 1000 km. Moreover, layer depths h_1 and h_2 are of the order 500 m each for large scale flows such as the Gulf stream at mid-latitude in the Atlantic ocean. For the eddy viscosity 50 m^{2}s^{-1} in [31], the Ekman constant r is then of the order 10^{-5} s^{-1}. The number r is large when the eddy viscosity is taken to be large.

The ocean water mean density ρ_0 is about 1025 kgm^{-3} or 1.025 gcm^{-3}. ρ_1 and ρ_2 are in the vicinity of ρ_0. The density difference $\rho_2 - \rho_1 \approx 25$ kgm^{-3}. But it is this small density difference that, in turn, determines pressure differences and thereby drives the ocean circulation [39].

The mean wind forcing f, i.e. the deterministic part of the curl of the wind stress on the top fluid layer is usually taken as a stationary, or time-averaged and even also zonally averaged, sinusoidal function. For example [18],

$$f = \frac{2\pi\tau_0}{\rho_0 h_1 L} \sin\frac{2\pi y}{L},$$

where the wind tension τ_0 is of order 1 dyne/cm^2 or of order 0.1 N/m^2. With this mean wind forcing and physical parameters specified above, the inequality (38) turns into the estimate $\varepsilon_{\mathcal{L}} \leq 1.35 \times 10^{-3}$ m. Therefore, in the case of Examples 7.1 or 7.2 for the number N of functionals we obtain the estimate $N \approx 10^{18}$. Thus, Theorem 5.1 should only be considered as a qualitative assertion about finite-dimensionality of the long-time behavior of the stochastically forced two-layer quasi-geostrophic fluid system. As for Theorem 6.1, similar calculations show that the condition (45) can be valid under some special choice of parameters.

8. Summary

We have considered asymptotic probabilistic dynamics of the stochastically forced two-layer quasigeostrophic fluid system. We have proved that the stochastic two-layer fluid system has finite sets of asymptotically determining functionals in probability, such as determining modes and determining nodes; see Theorem 5.1. We have further shown that the asymptotic probabilistic dynamics of this system depends only on the top fluid layer, provided that the Wiener process and the fluid parameters satisfy a certain condition, i.e., the inequality (42); see Theorem 6.1. In particular, this latter condition is satisfied when the trace of the covariance operator of the Wiener process is small enough (see Corollary 6.1) and the Ekman constant r is sufficiently large (see the inequality (45)). Note that the generalized time derivative of the Wiener process models the fluctuating part of the wind stress forcing on the top fluid layer, and the Ekman constant r measures the rate for vorticity decay due to the friction in the bottom Ekman layer.

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