Cavity Radiation, Photoelectric Effect, and Compton Effect

Physics 224: 25 March 2010
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Microscopic Units

Coulomb's Law (SI):

$$V(r) = k \, \frac{q_1 \, q_2}{r}$$

where $k = 8.988 \times 10^9$ J m /C^2,

Let us use the parameters in appropriate units:

- q_1, q_2: Coulombs (C)
- r: meters (m)
- V: Joules (J)

Note: $rV(r)$ can be expressed in units of J m. Fundamental unit of charge: $e_0 = 1.602 \times 10^{-19}$ C. Thus

$$k \, e_0^2 = 8.988 \times 10^9 \times (1.602 \times 10^{-19})^2 \text{ J m}$$

Microscopic unit of energy: electron Volt (eV)

$$1 \text{ eV} = e_0 \times 1 \text{ Volt} = 1.602 \times 10^{-19} \text{ J}$$

or

$$k \, e_0^2 = 1.440 \times 10^{-9} \text{ eV m} = 1.440 \text{ eV nm}$$

Velocity of light

$$c = 2.998 \times 10^8 \text{ m/s} = 2.998 \times 10^{17} \text{ nm/s}$$

Cavity Radiation

Stefan-Boltzmann Law (Universal):

$$I(T) = \sigma T^4$$

where $I(T)$ is the flux per unit area (in Watts/m^2) of electromagnetic radiation passing out of a small hole in a cavity that is held inside at Kelvin temperature T and

$$\sigma = 5.670 \times 10^{-8} \, \frac{\mathrm{W}}{\mathrm{m}^2 \, \mathrm{K}^4}$$

The radiation from human body (treated as a ideal radiator) (internal temperature: 310 K and surface area of order $1\,\mathrm{m}^2$) is about 520 W, and the thermal output is an average of about 100 W. (2000 kcal/day). For thermal equilibrium of a person inside a tub, the "equilibrium temperature" is about 293 K.[1]

Energy per unit volume inside cavity $u(T)$ can be determined.

$$I(T) = u(T) \times \frac{c}{4}$$

Energy per unit volume per unit frequency inside cavity $u(f, T)$ satisfies

$$I(f, T) = u(f, T) \times \frac{c}{4}$$

The factor $c/4$ arises from integration over a small neighborhood of the hole, because only particles very near the hole and traveling in the proper direction escape in an infinitesimal time interval. Note that

$$\int_0^\infty u(f, T) \, df = u(T)$$

Wien Displacement Law:

$$u(f, T) = \frac{8\pi \, f^3}{c^3} \, F(f/T)$$

[1] Why do we "shiver" when submerged in water that is below a temperature of about 20 °C? Why is a hot tub "too hot" above 40 °C?

where F is a function of the ratio f/T.[2] The peak wavelength λ_{max} at temperature T satisfies the empirical relation

$$\lambda_{max}\, T = 3 \times 10^{-3} \text{ m} \quad \text{K}$$

Inside the (hot) cavity at visible wavelengths, it is a very good fit to data to use

$$F(f/T) \approx \kappa\, e^{-\alpha f/T}$$

with suitable parameters κ and α.

This frequency distribution is independently measurable, and very important whenever light passes through or is reflected from matter.[3]

1900: Planck's quantum hypothesis:

Planck obtained the formula

$$F(f/T) = h/\left(e^{h\,f/(k\,T)} - 1\right) \tag{1}$$

which reduced to the Wien formula for "high frequencies", $h\,f \gg k\,T$. It also reproduced the low-frequency spectrum of cavity radiation, for the regime $h\,f \ll k\,T$, as well as in the transitional region in which these two quantities are of the same order of magnitude. Planck's formula was obtained by a remarkable assumption concerning the electromagnetic radiation inside the cavity:

> Radiation inside cavity consists of quanta of electromagnetic radiation (photons), where the energy of a single photon is
>
> $$E = h\,f$$
>
> where f is the frequency in cycles per second (Hz) and h is Planck's constant
>
> $$h = 6.626 \times 10^{-34} \text{ J sec} = 4.136 \times 10^{-15} \text{ eV sec}$$

[2]The name refers to the fact that, if f_{max} is defined as the maxima frequency in the spectrum of electromagnetic radiation inside a cavity held at temperature T, that "frequency peak" varies with temperature so that f_{max}/T does not change.

[3]Why do dogs (and people) swelter inside cars on hot summer days, when it is actually rather pleasant outside?

Planck asserted that each of the modes of electromagnetic radiation inside a cavity contains n quanta (photons), where $n = 0, 1, 2, \ldots$, so that the energy of the n photon state is $n\,h\,f$. The relative probability of occupation of the n photon state is

$$P_m = \exp\left(\frac{-n\,h\,f}{k\,T}\right)$$

The Planck formula (1) follows, expressed in terms of a single parameter, h. It was a remarkable triumph, but nobody in the world understood why it worked – including Planck himself.[4]

One may express the photon energy in terms of the wavelength of light $\lambda = c/f$:

$$\begin{aligned} E &= \frac{h\,c}{\lambda} \\ \text{where} \quad h\,c &= (4.136 \times 10^{-15}) \times (2.998 \times 10^{17}) = 1240 \ \text{ eV nm} \\ E(\text{eV}) &= \frac{1240}{\lambda\,(\text{nm})} \end{aligned}$$

Visible light $400\,\text{nm} \le \lambda \le 700$ nm, or $E \approx 2 - 3$ eV for a single photon.

Photoelectric Effect

"Accidental Discovery": Hertz 1896

Light strikes a metallic surface, and electrons come out of the material. Distinctive features observed:

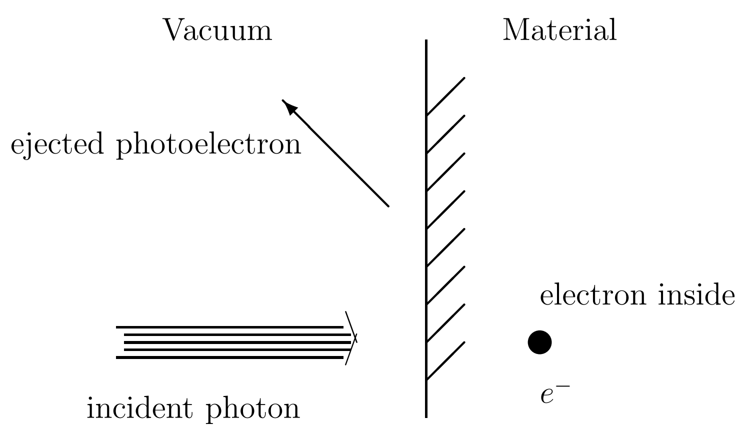
- "Cold" emission (not thermionic or "hot" emission)
- Light comes out almost instantaneously (local heating?)
- Blue light more likely to produce photoelectrons than red. (Why the frequency dependence?)

[4] Planck's quantum hypothesis was denounced by a number of outstanding research scientists (including Lord Rayleigh). Most physicists thought that he was "bonkers".

- Some metals work better than others, but NOT because they are better conductors of electricity. (In fact, the more chemically active metals seem to work better.) Why?

- The effect is difficult to reproduce.

  (the metallic surfaces must be VERY VERY clean. Why?



Photoelectric Effect

Tentative Explanation: Einstein (1905)[5]

- "Billiard ball" collision between one electron and one quantum of light – "photon".

- Kinetic energy of incident photon equals the kinetic energy of the emitted photo-electron $h\,f$, subtracted by the energy lost by the electron in escaping from the material. The maximum kinetic energy of the electron is called K_{max}, and the minimum energy that can be lost in getting out of the material, W, the work function of the material in question.

- According to Einstein,

$$K_{max} = h\,f - W \tag{2}$$

[5] Albert Einstein was "Mr. Nobody" when he published this result.

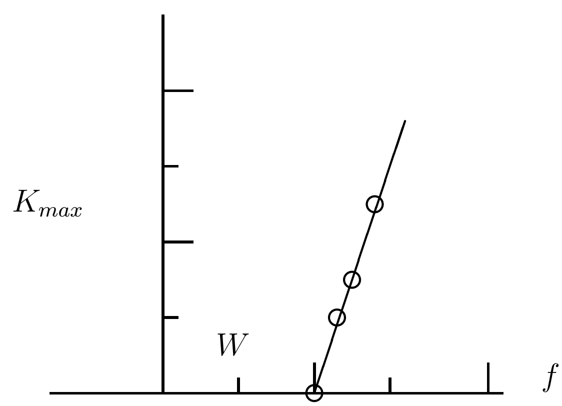
At a given frequency f, measure K_{max}. Note that K_{max} changes with frequency.

"stopping potential" in Volts; multiply by e_0 to convert to eV.

Typical Data:

λ (nm)	$f = c/\lambda$ (Hz)	K_{max} (eV)
300	10.0×10^{14}	1.9
400	7.5×10^{14}	0.9
500	6.0×10^{14}	0.4
580	5.2×10^{14}	0.
600	5.0×10^{14}	–

According to Eq.(2), a graph of f versus K_{max} should be a straight line of slope $1/h$ and intercept ($W = 0$) with the abscissa, $f = f_0$. That is, the formula $W = h\, f_0$, as shown



K_{max} vs. f

For this case $k_{max} = 0$ when $f = 5.2 \times 10^{14}$ Hz, so that[6]

$$h\, c = W \lambda_{min} = (2.1) \times 580 \approx 1280\, \mathrm{eV\, sec}$$

[6]Close, but no cigar!

While the theoretical result is conceptually simple if you happen to believe in photons as little pieces of light[7], its experimental verification was exceedingly difficult at the time. Why? Millikan (1913) was able to confirm the Einstein formula.

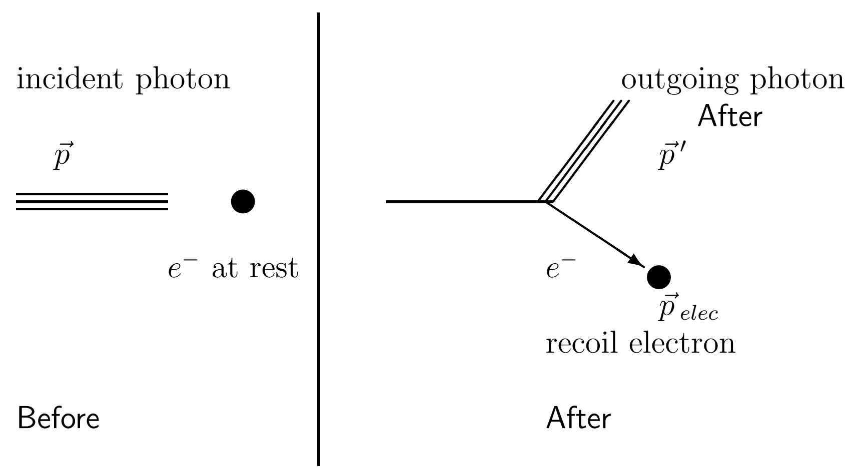
Einstein's explanation of the photoelectric effect assumes that a single photon collides with a single electron in the material. However, many many photons hit the material surface (of order 10^{18} per second in a one Watt source of visible light), it is essentially impossible to detect a single "billiard ball" collision.

How do you get around this serious obstacle?

SIMPLE: just use x-rays, in which a individual photon has a few keV of electrical energy, and can produce an image on a cathode ray tube.

Compton Effect (1921)

Billiard ball collision between a single electron and a single photon:



Compton Effect

The (Relativistic) Momentum $\vec{p}$ and energy E for a particle may be expressed in terms of its rest mass m and velocity $\vec{v}$.

[7]Otherwise, it would be sheer nonsense.

$$
\begin{aligned}
E &= \frac{m\,c^2}{\sqrt{1 - v^2/c^2}} \\
\vec{p} &= \frac{m\,\vec{v}}{\sqrt{1 - v^2/c^2}}
\end{aligned}
$$

The equivalent inverse relations are

$$
\begin{aligned}
E^2 &= c^2\,p^2 + m^2\,c^4 \\
\vec{v} &= \frac{c^2\vec{p}}{E}
\end{aligned}
$$

Although the direct relations have an indeterminant form 0/0 in the simultaneous limit

$$
m \to 0 \qquad \text{as well as} \qquad |v| \to c
$$

the inverse relations are well-defined in the massless limit $m = 0$; namely

$$
\begin{aligned}
E &= c\,p \\
|\vec{v}\,| &= c
\end{aligned}
$$

Thus, for the electron-photon collision we have

$$
E_{elec} = \sqrt{c^2\,p_{elec}^2 + m^2\,c^4} \qquad \text{and} \qquad E_\gamma = c\,p_\gamma
$$

According to Onkel Albert, the total (relativistic) momentum and energy must both be conserved in any collisional process ... no matter what the dynamical details may happen to be.

Thus for our collision, we require they must be the same before and after the collision:

$$
\begin{aligned}
E + m\,c^2 &= E' + E_{elec} \\
\vec{p} &= \vec{p}\,' + \vec{p}_e
\end{aligned}
$$

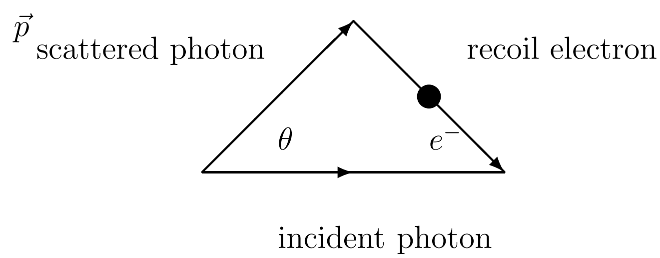
It is extremely difficult to see the recoil electon, which was initially at rest in the material – one of many in that category. However, one can detect a beam of x-rays moving in a particular direction. Thus, it is kinematically wise to eliminate both the energy and the momentum of that recoil electron, using the formula

$$m^2\, c^4 = E_{elec}^2 - c^2\, p_{elec}^2 \tag{3}$$

We cast the conservation equations into the form

$$\begin{aligned} E_{elec} &= E - E' + m^2\, c^4 \\ \vec{p}_{elec} &= \vec{p} - \vec{p}\,' \end{aligned} \tag{4}$$

The second relation among vector quantities may be visualized in terms of the Momentum Triangle:



Momentum Triangle

Since a triangle can always be embedded in some planar surface, we can and will use the Law of Cosines to relate the length of the side, p_{elec} opposite to the inside angle θ to the lengths of the adjacent sides, p and p':

$$p_{elec}^2 = p^2 + p^{'\,2} - 2p\,p'\,\cos\theta \tag{5}$$

We apply (5), as well as (4), to express (3) in the following form:

$$m^2 c^2 = (p - p' + m\,c)^2 - p^2 - p^{'\,2} + 2p\,p'\,\cos\theta$$

Equivalently:

$$m\,c\,(p - p') = p\,p'(1 - \cos\theta)$$

or

$$\frac{1}{p'} - \frac{1}{p} = \frac{1}{m\,c}\,(1 - \cos\theta)$$

Since photons are massless, we get the following equivalent relations the incoming photon energy ($E = cp$) and that of the outgoing photon traveling at angle θ to the incident direction ($E' = cp'$):

$$\frac{1}{m\,c^2}\,(1 - \cos\theta) = \frac{1}{E'} - \frac{1}{E} \tag{6}$$

We multiply this formula by the factor $h\,c$ to convert the energy (in eV) into the wavelengths λ and λ' of the incoming and outgoing photons, respectively, to obtain the celebrated Compton equation

$$\lambda' - \lambda = \frac{h}{mc}\,(1 - \cos\theta) \tag{7}$$

According to this formula, the outgoing photon is deflected into some angle θ lose energy – that their final energy increases with θ