Electric Field of an Electric Dipole
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Let us begin by placing a charge $+q$ on the z-axis at position $(0,0,d/2)$ and a charge $-q$ symmetrically with respect to the origin at position $(0,0,-d/2$. Let us evaluate the electric fields at position $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. We obtain

$$\begin{aligned} \vec{E} &= \frac{kq\,\vec{r}_1}{r_1^3} - \frac{kq\,\vec{r}_2}{r_2^3} \\ \vec{r}_1 &= \vec{r} - \hat{k}\,d/2 \\ \vec{r}_2 &= \vec{r} + \hat{k}\,d/2 \end{aligned}$$

Consequently

$$\vec{E} = kq\,\vec{r}_1\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) - \frac{kq}{r_2^3}\,(\vec{r}_2 - \vec{r}_1)$$

We use the identities

$$\begin{aligned} r_1^2 &= r^2 - zd + d^2/4 \\ r_2^2 &= r^2 + zd + d^2/4 \\ \vec{r}_2 - \vec{r}_1 &= d\,\hat{k} \end{aligned}$$

When $d = 0$ both terms in the formula for $\vec{E}$ are zero. We collect terms of order d in each term to obtain

$$\begin{aligned} \vec{E} &= kq\,\frac{\vec{r}}{r^3}\left(1 + \frac{3dz}{2r^2} - 1 + \frac{3dz}{2r^2}\right) - kq\frac{d\,\hat{k}}{r^3} \\ &= \frac{3k\,\vec{r}\,(qdz)}{r^5} - \frac{k(qd\,\hat{k})}{r^3} \end{aligned}$$

We express this result in terms of the dipole moment $\vec{p} = qd\,\hat{k}$ as follows:

$$\vec{E} = \frac{3k\,\vec{r}\,(\vec{p}\cdot\vec{r})}{r^5} - \frac{k\,\vec{p}}{r^3}$$

If another dipole of dipole moment $\vec{p}'$ is placed at a distance $\vec{r}$ from this dipole, the potential energy of interaction is

$$U = -\vec{p}'\cdot\vec{E} = \frac{k}{r^3}\left(\vec{p}\cdot\vec{p}' - \frac{3(\vec{p}\cdot\vec{r})\,(\vec{p}'\cdot\vec{r})}{r^2}\right)$$

The potential energy is unchanged on switching the dipole moments: $\vec{p} \leftrightarrow \vec{p}'$.