Physics 221

Quizes and Examinations

Fall 2006

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"Physics can only be learned by thinking, writing, and worrying."
-David Atkinson and Porter Johnson (2002)

"There is no royal road to geometry."
-Euclid

1. PHYS 221 - 005, QUIZ 1, 31 August 2006

   A block of mass $M = 5kg$, initially at rest on a horizontal table, is attached to a rigid support by a spring with spring constant $k = 5000N/m$. A bullet of mass $m = 5$ grams and speed of $v = 1000$ meters per second strikes and is imbedded in the block. Assume that the compression of the spring is negligible until the bullet is imbedded, and determine the following:

   - the speed of the block immediately after the collision.
   - the maximum compression of the spring.
   - the period of the resulting simple harmonic motion.

   <u>Solution</u>:

   During the collision mechanical energy is converted into heat, whereas momentum is conserved. Thus, if the final speed of bullet plus block is V, then

$$m\,v = (M+m)\,V$$

   or

$$V = mv/(M+m) = (5/5005)\cdot 1000 \approx 1\ m/s$$

   If the maximum compression of the spring is d, then the kinetic energy of block plus bullet just after the collision is equal to the compressional energy at maximum compression:

$$\begin{aligned} \frac{1}{2}k\,d^2 &= \frac{1}{2}(M+m)V^2 \\ d^2 &= (M+m)V^2/k = (5.005)(1)^2/(5000) \approx 0.001 \\ d &\approx 0.03\,m \end{aligned}$$

   From Newton's second law for the mass plus string, the displacement of the spring, $x(t)$, satisfies the differential equation

$$(M+m)\,\ddot{x} = F = -kx$$

   or

$$a = -\frac{k}{M+m}\,x = -\omega^2\,x$$

so that $\omega^2 = k/(M+m) = 5000/(5.005) \approx 1000$. Thus $\omega \approx 32\ rad/sec$, corresponding to a period $T = 2\pi/\omega \approx 0.2\ sec$.

This is the basis for the **ballistic pendulum**, which occasionally appeared in general physics laboratories several decades ago. It was also used by forensic specialists to measure muzzle speeds. What do they do now?

2. PHYS 221 - 006, QUIZ 1, 01 September 2006

   A loudspeaker diaphragm is oscillating in simple harmonic motion with a frequency of $f = 500\ Hz$ and a maximum displacement of $A = 1.0\ mm$. Determine the following:

   - the angular frequency of vibration.
   - the maximum speed of the diaphragm.
   - the maximum acceleration of the diaphragm.

   Solution:

   Simple harmonic motion:

$$\begin{aligned} \omega &= 2\pi f = 1000\,\pi \approx 3100\ rad/sec \\ x &= A \sin \omega t \\ v = \frac{dx}{dt} &= \omega A \cos \omega t \\ a = \frac{dv}{dt} &= -\omega^2 A \sin \omega t \end{aligned}$$

   Thus

$$\begin{aligned} v_{max} &= \omega A \approx 3100 \cdot (0.001) = 3.1\ m/s \\ a_{max} &= \omega^2 A \approx (3100)^2 \cdot (0.001) \approx 9800\ m/s^2 \end{aligned}$$

   Note: The maximum acceleration corresponds to $1000\,g$, and the diaphragm will surely be ripped to pieces. Why is vibration at high frequency more damaging than at low frequency?

3. PHYS 221 - 003, QUIZ 1, 07 September 2006

   The speed of a transverse wave on a string is $v_T = 200$ meters per second, when the string tension is $T = 150$ Newtons.

- Determine the mass per unit length μ of the string.
- What tension T' is required in the string to raise the wave speed to $v' = 250$ meters per second?

Solution:

The speed of transverse vibrations is $v = \sqrt{T/\mu}$, so that the mass per unit length is

$$\mu = T/v^2 = (150)/(200)^2 = 3/800 \approx 0.0038\, kg/m$$

A string of mass per unit length μ and tension T has transverse speed $v = \sqrt{T/\mu}$.

The same string, with tension T', has transverse vibration speed $v^{'} = \sqrt{T'/\mu}$. Thus

$$\frac{v^{'}}{v} = \sqrt{\frac{T'}{T}}$$

$$T' = T\left(\frac{v^{'}}{v}\right)^2 = 150\left(\frac{250}{200}\right)^2 \approx 230\, N$$

Ergo $\cdots$ the greater the tension the higher the pitch. You can check it out on a guitar – even a toy guitar.

4. PHYS 221 - 004, QUIZ 1, 12 September 2006

A string of length $L = 1.5$ meters has a mass of $m = 4.0$ grams. It is stretched between fixed supports with a tension of $T = 25$ Newtons.

- Determine the speed of transverse vibrations of the string.
- What is the lowest resonant frequency of this string, in Hertz?

Solution:

The mass per unit length is

$$\mu = m/L = (0.004\, kg)/(1.5\, m) \approx 2.7 \times 10^{-3}\, kg/m$$

Thus the velocity is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{25}{2.7 \times 10^{-3}}} \approx \sqrt{9200} \approx 96\ m/s$$

For the lowest resonant mode, $\lambda = 2L = 3$ meters, and $f = v/\lambda \approx 96/3 = 32\ Hz$. This is an "elephant rumble".

5. PHYS 221 - 005, QUIZ 2, 14 September 2006

   A point source emits $P = 50$ watts of sound isotropically. A small microphone intercepts the sound in an area of $A_{mike} = 0.5\ cm^2$, a distance $R = 200$ meters from the source. Calculate the power P_{mike} intercepted by the microphone, in Watts, as well as the sound intensity I_{mike} at the microphone, in Watts per meter-squared and in deciBels.

   Solution:

   The isotropic source has intensity $I = P/(4\pi R^2)$ at a distance R, so that $I_{mike} = 50/(4\pi\ 200^2) \approx 9.95 \times 10^{-5} W/m^2$. The corresponding deciBel reading is

$$\begin{aligned} \beta &= 10 \cdot \log_{10}\left(\frac{I_{mike}}{I_{threshold}}\right) \\ &\approx 10 \cdot \log_{10}\left(\frac{9.95 \times 10^{-5}}{10^{-12}}\right) \\ &= 10 \cdot \log_{10}(9.95 \times 10^{7}) = 10 \cdot (7.9097) \approx 79\ dB. \end{aligned}$$

   The sound is thus loud, but not painful. Recall that a **hundred Watt amplifier** requires $100W$ of electric power, but converts only a small amount of it into acoustic energy.

   The power into the microphone is

$$P_{mike} = I_{mike}\, A_{mike} = (9.95 \times 10^{-5}) \cdot (5 \times 10^{-4}) \approx 5 \times 10^{-9}\ W$$

   A few nanoWatts, thus. Be sure to stand close to the microphone when you speak!

6. PHYS 221 - 006, QUIZ 2, 15 September 2006

   A pipe $L = 1.0$ meters long and closed at one end is filled with an unknown gas. The third lowest harmonic frequency is measured to be $f = 600\,Hz$.

   - What is the speed of sound v in the unknown gas?
   - What is the fundamental frequency f_0 for this pipe when it is filled with the unknown gas?

   Solution:

   For the pipe of length L with one end closed the fundamental mode corresponds to a quarter wavelength; $\lambda = L/4$. (a node at closed end, and an antinode at the open end). The resonant wavelengths are $L = (2n + l)\lambda/4$, for $n = 0, 1, 2, \cdots$.

   The third wavelength, $n = 2$, gives $l = 5\lambda_2/4$, so that $\lambda_2 = 4L/5 = 0.8\,m$. This mode has a frequency $f_2 = 600\,Hz$, so that the speed of sound inside the pipe is $v = \lambda f = 0.8 \cdot 600 = 480\,m/s$.

   For the fundamental mode

$$\begin{aligned}\lambda_o &= 4L = 4\,m \\ f_o &= v/\lambda_o = 480/4 = 120\,Hz\end{aligned}$$

   The speed of sound on a gas depends upon the atom / molecule involved, as well as the temperature. What gas is in this tube?

7. PHYS 221 - 003, QUIZ 2, 21 September 2006

   Two equal positive charges, $+Q$, are placed on opposite sides of a square of side a. Two equal negative charges, $-q$, are placed on the two other sides. Determine the ratio Q/q, if the net force on each of the positive charges $+Q$ is zero.

   Solution:

   Put the postive charges $+Q$ at locations $(0,0)$ and (a,a), with the negative charges $-q$ at locations $(a,0)$ and $(0,a)$.

   The forces on the charge at the origin come from the other three charges. The charge $+Q$ lies at distance $\sqrt{2}a$, so that the magnitude of the force $\vec{F}_2$ is $kQ^2/(2a^2)$. The two negative charges, which lie at distance a from that charge, produce forces $\vec{F}_1$ and $\vec{F}_3$, respectively. They are each of magnitude kQq/a^2. We separate these forces into components as indicated below:

Location	Force		x-component	y-component
$(a,0)$	$\vec{F}_1$	|	$\frac{kQq}{a^2}$	0
(a,a)	$\vec{F}_2$	|	$-\frac{kQ^2}{2\sqrt{2}a^2}$	$-\frac{kQ^2}{2\sqrt{2}a^2}$
$(0,a)$	$\vec{F}_3$	|	0	$\frac{kQq}{a^2}$
	$\vec{F}_{total}$	|	0	0

For the net force on the charge at the origin to be zero, we must have

$$\frac{kQq}{a^2} = \frac{kQ^2}{2\sqrt{2}a^2}$$

or $q = Q/(2\sqrt{2})$.

The net electric field on the charge at (a,a) also vanishes, by symmetry.

Note: The forces on the negative charges do not vanish. They go flying away from one another in a diagonal direction – never to return! The net electric force on the negative charge at $(0,a)$ is $(\sqrt{2}-1)\cdot kQ^2/(2a^2)$.

8. PHYS 221 - 004, QUIZ 2, 26 September 2006

Two positive charges, each of magnitude $Q = 10^{-6}$ Coulombs are placed symmetrically on opposite sides of an insulating wire, each a distance $d = 1$ meter from the wire along a line perpendicular to the wire. A negative charge $q = -10^{-6}$ Coulombs is imbedded on a bead that slides smoothly along the wire. The mass of the bead is $m = 0.001\ kg$.

- If the bead lies a (small) distance x to the right of the center line joining the positive charges, determine the net force on the bead.
- Determine the frequency f of small oscillations of the bead about the line joining the positive charges.

Solution:

The force of attraction of the bead (negative charge q) and each positive charge $+Q$ is of magnitude

$$F_o = \frac{kQ\,|q|}{r^2} = \frac{kQ\,|q|}{d^2+x^2}$$

Each force points toward the positive charge in question. The vertial component of the total force on the bead is zero, by "up-down symmetry".

The net force on the bead is thus horizontal, and of magnitude $2F_o \cos\theta$, where θ is the angle between a force direction and the $-x$-direction. The angle θ occurs in a right triangle with adjacent side x, opposite side d, and hypoteneuse is $r = \sqrt{d^2 + x^2}$. Thus,

$$\cos\theta = x/r = \frac{x}{\sqrt{d^2 + x^2}}$$

When x is extremely small in comparison to d; that is, $x \ll d$; we may drop x in comparison to d in the denominator to obtain $r \approx d$ and $\cos\theta \approx x/d$. Thus the horizontal force on the bead is (approximately)

$$F_x = -2F_o \cos\theta \approx -2 \cdot \frac{kQ\,|q|}{d^2} \cdot \frac{x}{d} = \frac{2kQ\,|q|}{d^3}\,x$$

The direction of the resultant force is to the left; that is, the $-x$ direction.

According to Newton's second law, $\vec{F} = m\vec{a}$, the bead should accelerate to the left when x is positive, with x-component

$$a_x = F_x/m = -\frac{2kQ\,|q|}{md^3}\,x$$

Since the restoring force is opposite in direction to the displacement, this corresponds to simple harmonic motion, as seen from the relation $a = -\omega^2 x$. We therefore obtain

$$\omega^2 = \frac{2kQ\,|q|}{md^3}$$

By putting in the numbers, we obtain

$$F_o = \frac{kQ\,|q|}{d^2} \approx \frac{(9 \times 10^9) \cdot 10^{-6} \cdot 10^{-6}}{(1)^2} = 9 \times 10^{-3}\,N$$

Thus

$$F_x \approx -2 \cdot (9 \times 10^{-3}\,N)\frac{x}{1\,mt} = -1.8 \times 10^{-2}\,\frac{N}{mt} \cdot x(mt)$$

Thus, "effective spring constant" is present; $k_{eff} \approx 1.8 \times 10^{-2} N/mt$. We compute the corresponding angular velocity:

$$\begin{aligned} \omega^2 &= k_{eff}/m \approx 18 \\ \omega &\approx 4.2\,rad/sec \\ f = \omega/(2\pi) &\approx 0.7\,Hz \\ T = 1/f &\approx 1.4\,sec \end{aligned}$$

Could this vibration be used to create an electric clock? Why or why not?

9. PHYS 221 - 005 QUIZ 3 28 September 2006

A ring of radius R contains a total charge q, distributed uniformly along it. A second ring, placed concentrically and in same plane as the first ring, is of radius $2R$, and it contains total charge Q, again distributed uniformly along it. The electric field along the axis of the ring at a distance R from its center is measured and found to be zero. Determine the charge Q on the second ring. You may express your answer in terms of q and R.

Solution:

We begin by considering the electric field produced by the inner ring, which contains a charge q spread uniformly over its circumference, a circle of radius r, along the axis of symmetry and a distance z from the center of the circle. The electric field E_1 lies along the symmetry axis. We may calculate that component E_{1z} by integrating over the charge distribution:

$$E_{1z} = \int dE_{1z} = k \int \frac{dq}{r} \cos\theta$$

The distance $r = \sqrt{R^2 + z^2}$, and $\cos\theta = z/r = z/\sqrt{R^2+z^2}$. As a consequence

$$E_{1z} = k \int \frac{dq}{\sqrt{R^2+z^2}} \frac{z}{\sqrt{R^2+z^2}} = \frac{kqz}{(R^2+z^2)^{3/2}}$$

The field from the second ring is obtained by making the replacements $q \rightarrow Q$ and $R \rightarrow 2R$, to obtain

$$E_{2z} == \frac{kQz}{(4R^2+z^2)^{3/2}}$$

The total field on axis is

$$E_z = E_{1z} + E_{2z} = \frac{kqz}{(R^2+z^2)^{3/2}} + \frac{kQz}{(4R^2+z^2)^{3/2}}$$

That field vanishes at the point $z = R$, under the condition

$$\frac{kqR}{2\sqrt{2}R^3} + \frac{kQR}{5\sqrt{5}R^3} = 0$$

or

$$Q = -\frac{5\sqrt{5}}{2\sqrt{2}}q$$

The charges on the ring must be of opposite signs for the field to vanish at the point in question.

10. PHYS 221 - 006 QUIZ 3 29 September 2006

Three particles, each with positive charge Q, form an equilateral triangle, with the sides of length a. What is the magnitude and direction of the electric field produced by the particles at the midpoint of a particular side?

Solution:

Let us draw the equilateral triangle with a a horizontal base, and take the point in question on the base. There are two charges on that base, at distances $a/2$ from the that point. Their fields are each of magnitude $kQ/(a/2)^2$, and lie in opposite directions, so that they cancel. In addition, there is a charge Q at a height $h = \sqrt{a^2 - (a/2)^2} = \sqrt{3}a/2$ above the point, which produces a downward electric field

$$E_z - \frac{kQ}{h^2} = -\frac{4kQ}{3a^2}$$

Thus, the net electric field is downward, with that downward component.

11. PHYS 221 - 003 QUIZ 3 05 October 2006

A thin (insulating) disk of radius R contains a total charge Q imbedded uniformly into its surface.

- Determine the magnitude and the direction of the electric field at points along the axis of the disk, at distance z from its axis.
- A second identical disk contains total charge $-Q$ that is imbedded uniformly into its surface. It is placed coaxially with the first disc, with the two disks lying in parallel planes. The separation of these two discs is $2R$.

- Determine the magnitude and direction of the electric field at a point halfway between the disks, along the axis of symmetry.

Solution:

Let us begin by calculating the electric field on axis at a distance z from the first disk. It lies along that axis, by symmetry. Let us divide the disk into thin concentric "onion rings" of radius r, which contain a total charge dq. Since the charge density on the disk is $\sigma = Q/(\pi R^2)$, the charge in a ring is $dq = \sigma \cdot dS$. where dS is area of a ring of radius r and thickness dr; $dS = 2\pi r dr$. Thus, $dq = 2Qrdr/R^2$, and the axial component of the electric field produced by hat ring is

$$dE_z = dE\cos\theta = \frac{kdq}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} = \frac{kzdq}{(r^2+z^2)^{3/2}} = \frac{2kQz}{R^2}\frac{rdr}{(r^2+z^2)^{3/2}}$$

The total electric field may be found by integrating over r:

$$\begin{aligned} E_z &= \frac{2kQz}{R^2}\int_0^R \frac{rdr}{(r^2+z^2)^{3/2}} = \frac{2kQz}{R^2}\int_0^R d\left(-\frac{1}{\sqrt{r^2+z^2}}\right) \\ &= \frac{2kQz}{R^2}\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right) = \frac{2kQ}{R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right) \\ &= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right) \end{aligned}$$

We set $z = R$ to obtain the electric field from the first disk:

$$E_{1z} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{2}}\right)$$

The second disk is negatively charged, and the same distance from the point in question, so that the total field from the two disks is twice this value:

$$E_z = \frac{\sigma}{\varepsilon_0}\left(1-\frac{1}{\sqrt{2}}\right)$$

12. PHYS 221 - 004 QUIZ 3 10 October 2006

A point charge $+Q$ is placed inside a thin hollow isolated conducting spherical shell of radius R — at the center of that shell, as shown. A total charge

$-2Q$ is placed on that conducting shell, as well. Determine the magnitude and direction of the electric field everywhere inside and outside the shell.

Solution:

Let us apply Gauss's Law to an arbitrary concentric sphere of radius r:

$$\varepsilon_0 \oint \vec{E} \cdot \vec{dS} = Q_{enc}$$

Because of spherical symmetry, the electric field on that spherical surface has only a radial field E_r, which is independent of location on the surface. Thus

$$\begin{aligned} \varepsilon_0 E_r \oint dS &= Q_{enc} \\ \varepsilon_0 E_r 4\pi r^2 &= Q_{enc} \\ E_r &= \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{kQ_{enc}}{r^2} \end{aligned}$$

The only charge inside the conducting shell ($r < R$) is a point charge $+Q$ at the center, so that the electric field is radially outward and of magnitude

$$E_r = \frac{kQ}{r^2}$$

Eithin the conductor, the electric field must vanish. Consequently, there must be net charge $-Q$ distributed uniformly ovr the inner surface of the conductor. In addition, a charge $-Q$ must be distributed uniformly over the outer surface of the conductor. For $r > R$ the net charge inside the gaussian surface is $Q - 2Q = -Q$. Thus the electric field there is radially inward:

$$E_r = -\frac{kQ}{r^2}$$

13. PHYS 221 - 005 Quiz 4 12 October 2006

Two identical solid, insulating spheres of radius R contain uniformly distributed charge throughout their interiors. The sphere on the left contains total charge Q_1, whereas the right sphere contains total charge Q_2. The spheres are placed tangentially at the surface, with their edges touching at a point. Draw the line joining the centers of the spheres. We observe that the electric field vanishes along that line, halfway from the center of the left sphere, and toward the right sphere. Determine the ratio Q_1/Q_2.

Solution:

The location in question lies inside the left sphere, at a distance $r = R/2$ from its center, as well as outside the right sphere, at a distance $r = 3R/2$ from its center. Accordingly, the electric field E_1 arising from the charge in the left sphere it directed toward the right. Its magnitude can be computed from Gauss/s Law:

$$E_1 = \frac{kQ_{enc}}{r^2} = \frac{kQ_1}{r^2}\frac{r^3}{R^3} = \frac{kQ_1 r}{R^2} = \frac{kQ_1}{2R^2}$$

The electric field E_2 arising from the charge in the right sphere is directed to the left, and its magnitude is given by

$$E_2 = \frac{kQ_{enc}}{r^2} = \frac{kQ_2}{r^2} = \frac{4kQ_2}{9R^2}$$

We thus have

$$\begin{aligned} E_1 &= E_2 \\ \frac{kQ_1}{2R^2} &= \frac{4kQ_2}{9R^2} \\ Q_1 &= \frac{8}{9}Q_2 \end{aligned}$$

Note that the charges on the two spheres are of the same sign, and nearly equal in magnitude.

14. PHYS 221 - 006 QUIZ 4 13 October 2006

A charged particle of charge Q is attached in place at the center of a thick conducting spherical shell. There is no net charge upon the shell itself. The inner radius of the shell is a, and its outer radius is b, as shown.

Determine the magnitude and direction of the electric field in each of these three regions:

- $r < a$: inside the shell
- $a < r < b$: within the shell
- $r > b$ outside the shell

Solution:

This is a problem involving spherical symmetry. Accordingly, we choose Gaussian surfaces that are spheres, centered at the center of the shell. The electric field for always lies in a radial direction; let its radial component be E_r.For any such Gaussian surface, we apply Gauss's law to obtain:

$$\begin{aligned} \varepsilon_0 \oint \vec{E} \cdot \vec{dS} &= Q_{enclosed} \\ \varepsilon_0 E_r (4\pi r^2) &= Q_{enclosed} \\ E_r = \frac{Q_{enclosed}}{4\pi\varepsilon_0\, r^2} &= \frac{k\, Q_{enclosed}}{r^2} \end{aligned}$$

For $r < a$, the only charge inside the Gaussian surface (sphere or radius r) is the point charge at the center, so that $Q_{enclosed} = Q$, and

$$E_r = \frac{k\,Q}{r^2}$$

For $a < r < b$, the surface of the sphere lies entirely within the conductor, and the (static) electric field is zero there. Thus $E_r = 0$ and $Q_{enclosed} = 0$. There must be a total charge $-Q$ on the inside surface of the shell, at $r = a$.

For $r > b$ the total charge enclosed in the sphere of radius r is Q, since the shell itself is electrically neutral. For this region we obtain

$$E_r = \frac{k\,Q}{r^2}$$

There must be a net charge $+Q$ on the outer surface of the conducting shell, to maintain its overall neutrality.

15. PHYS 221 - 004 QUIZ 4 24 October 2006

A 1.0 nanoFarad (nF) capacitor is charged to a potential difference of 20 Volts, and the charging battery is then disconnected.

This 1.0 nF capacitor is THEN connected in parallel with a second (initially uncharged) capacitor of unknown capacitance C. When equilibrium is reached, the potential difference across the plates of the 1.0 nF capacitor is then measured to be 5.0 Volts.

- What is the value of the unknown capacitance C, in nanoFarads?
- How much electrical energy (in Joules) is stored in the 1.0 nF capacitor just after the battery is disconnected?
- How much electrical energy is stored in each of the capacitors when equilibrium is reached?

Solution:

The charge that is initially on the 1 nF capacitor is

$$Q_0 = CV = 1.0 \times 10^{-9} \cdot 20 = 2.0 \times 10^{-8} C$$

The electrical energy initially stored in the capacitor is

$$\frac{1}{2} CV^2 = \frac{1}{2}(10^{-9})(20)^2 = 200\ nJ$$

After the connection, that capacitor has a potential of 20 V between the plates, and its final charge is

$$Q_1 = 1.0 \times 10^{-9} \cdot 5 = 0.5 \times 10^{-8} C$$

The charge on the second capacitor is thus $Q_2 = Q_0 - Q_1 = 1.5 \times 10^{-8} C$. Since that second capacitor also has a potential of 5 V between its places, Its capacitance is

$$C_2 = Q_2 / V_2 = 1.5 \times 10^{-8} / 5 = 3\ nF$$

The energies stored in the capacitors at equilibrium are

$$\begin{aligned} \mathcal{E}_1 &= \frac{1}{2} C_1 V_1^2 = \frac{1}{2}(10^{-9})(5)^2 = 12.5\ nJ \\ \mathcal{E}_2 &= \frac{1}{2} C_2 V_2^2 = \frac{1}{2}(3 \times 10^{-9})(5)^2 = 37.5\ nJ \end{aligned}$$

The total electrical energy at equilibrium is thus $50 nJ$. Consequently, $200 nJ$ of electrical energy has been dissipated as heat in this process.

16. PHYS 221 - 003 QUIZ 4 26 October 2006

A capacitor with unknown capacitance C and initial potential difference 40 Volts is discharged through a resistor of unknown resistance R, when a switch between them is closed at time $t = 0$ sec. At time $t = 5$ sec, the potential difference across the resistor is measured to be 15 Volts.

- What is the time constant of this RC circuit, in seconds?
- What is the potential difference across the capacitor at time $t = 15$ sec?

Solution:

The charge on the capacitor at time t is

$$Q(t) = Q_0\, e^{-t/\tau}$$

where τ is the time constant for the circuit. Since $Q = CV$, we may express the potential across the capacitor at time t in terms of the potential at $t = 0$ and the time constant τ:

$$\begin{aligned} V(t) &= V_0\, e^{-t/\tau} \\ 15 &= 40\, e^{-5/\tau} \\ 0.375 &= e^{-5/\tau} \\ -0.98 &= -\frac{5}{\tau} \\ \tau &= \frac{5}{.98} = 5.1\, sec \end{aligned}$$

The potential difference at $t = 15$ sec is

$$V(15) = 40 \cdot e^{-15/5.1} = 40 \cdot 0.053 = 2.1\, V$$

One may also write this result as

$$V(15) = 40 \cdot (15/40)^3 = 2.1\, V$$

17. PHYS 221 - 005 QUIZ 5 02 November 2006

In a capacitor of capacitance 800 nanoFarads, the charge gradually “leaks” from one plate to the other over the course of time. Suppose that the capacitor is initially charged to a potential of 30 Volts. After 24 hours, the potential across its plates has dropped to 20 Volts. By considering an appropriate *RC* circuit, determine the "equivalent resistance" between the plates, in Ohms.

Solution:

The charge on the capacitor at time t is given in terms of its charge Q_0 at time $t = 0$ and the time constant $t = RC$ as

$$Q(t) = Q_0\, e^{-t/RC}$$

Since $Q(t) = CV(t)$, where C is the capacitance, it follows that

$$\begin{aligned} V(t) &= V_0\, e^{-t/RC} \\ \frac{30}{20} = 1.5 &= e^{t/RC} \\ \ln 1.5 \approx 0.405 &= \frac{24 \cdot 3600\, sec}{RC} \\ R = \frac{8.64 \times 10^4}{0.405 \cdot (8 \times 10^{-7})} &\approx 2.66 \times 10^{11}\, \Omega \end{aligned}$$

One may obtain an approximate solution by saying that, in 24 hours, the charge leaked across the capacitor is

$$\Delta Q = C\Delta V = (8 \times 10^{-7}) \cdot 10 = 8 \times 10^{-6}\, C$$

Thus, the average current flowing during this time is

$$I = \frac{\Delta Q}{\Delta t} = \frac{8 \times 10^{-6}}{8.64 \times 10^4} = 9.2 \times 10^{-11}\, A$$

The resistance R is given by the average Voltage across the capacitor (25 Volts) divided by this current:

$$R = \frac{25}{9.2 \times 10^{-11}} = 2.7 \times 10^{11}\, \Omega$$

This resistance is high, because the capacitor remains charged for a long time.

18. PHYS 221 - 006 QUIZ 5 03 November 2006

Two wires, A and B, are made from different materials, with different cross-sectional areas. The wires are each cut to a length of 1 meter, an connected in series. A current of $3.0A$ passes through the wires.

Wire	Resistivity	Area
A	$1.0 \times 10^{-6}\ \Omega m$	$3.0 \times 10^{-4}\ m^2$
B	$2.0 \times 10^{-6}\ \Omega m$	$4.0 \times 10^{-4}\ m^2$

What is the potential drop (in Volts) across each wire? At what rate is electrical energy converted into heat (in Watts) in each wire?

<u>Solution</u>:

First, calculate the resistances of each wire:

$$R_A = \frac{\rho_A L_A}{S_A} = \frac{1.0 \times 10^{-6}\ 1}{3.0 \times 10^{-4}} = 0.0033\,\Omega$$
$$R_B = \frac{\rho_B L_B}{S_B} = \frac{2.0 \times 10^{-6}\ 1}{4.0 \times 10^{-4}} = 0.005\,\Omega$$

Thus the voltages across each wire are $V_A = R_A I = 0.01\,V$ and $V_B = R_B I = 0.015\,V$.

The power lost in each resistor can then be computed:

$$P_A = IV_A = 0.03\,W \cdots P_B = IV_B = 0.045\,W$$

19. PHYS 221 - 004 QUIZ 5 07 November 2006

A temperature-stable resistor is made by connecting a resistor made of Silicon in series with one made of Iron. If the required total resistance is 1000 Ohms, at temperatures around $20^o C$, what should the respective resistances of the Silicon and the Iron resistor be?

Note: Temperature coefficients of resistivity a (in K^{-1}):

- Iron: 6.5×10^{-3}
- Silicon: -70×10^{-3}

<u>Solution</u>:

At the reference temperature of $20^o C$, the resistances must add to $1000\,\Omega$:

$$R_{Fe} + R_{Si} = 1000\,\Omega$$

When the temperature changes by an amount ΔT, the individual resistances change, but the total resistance remains the same:

$$R_{Fe}(1+\alpha_{Fe}\,\Delta T)+R_{Si}(1+\alpha_{Si}\,\Delta T)=1000\,\Omega$$

As a consequence

$$\begin{aligned}\alpha_{Fe}\,R_{Fe}+\alpha_{Si}\,R_{Si} &= 0\\ 6.5\,R_{Fe}-70\,R_{Si} &= 0\\ R_{Fe} &= 10.8\,R_{Si}\end{aligned}$$

Thus we obtain

$$\begin{aligned}R_{Si} &= \frac{1000\,\Omega}{11.8}=85\,\Omega\\ R_{Fe} &= 1000\,\Omega-R_{Si}=915\,\Omega\end{aligned}$$

Notice that, because the (negative) temperature coefficient of resistivity of the semiconducting material silicon is much larger in magnitude than the (positive) temperature coefficient for iron, the resistance of the silicon resistor is much less than that of the iron resistor.

20. PHYS 221 - 003 QUIZ 5 09 November 2006

A flash Lamp L is placed across a capacitor of capacitance $C=0.20\ \mu F$. The combination is connected in series with a resistor of resistance $R=5\times10^4\ \Omega$ and a battery of Voltage 60 Volts. The switch is closed at time $t=0$. How much time elapses before the Voltage across the lamp reaches 40 Volts - at which point the lamp flashes briefly?

Note: The flash lamp does not conduct electricity at all until the Voltage across it reaches 40 Volts.

Solution:

The time constant for the circuit is

$$\tau=RC=(2.0\times10^{-7}\,F)\cdot(5\times10^4\,\Omega)=0.01\,sec$$

The charge on the capacitor is obtained from Kirchhoff's loop equation to be

$$Q(t)=C\mathcal{E}(1-e^{-t/\tau})$$

Thus the Voltage across the capacitor is

$$V(t) = Q(t)/C = \mathcal{E}(1 - e^{-t/\tau})$$

The battery voltage is $\mathcal{E} = 60\,V$, and the time at which the Voltage across the capacitor is $40\,V$ satisfies the relation

$$\begin{aligned} 40 &= 60(1 - e^{-t/(.01})\\ 1/3 &= e^{-t/(.01)}\\ 3 &= e^{t/\tau}\\ \ln 3 \approx 1.10 &= t/(.01)\\ t &\approx 1.1 \times 10^{-2}\,sec \end{aligned}$$

This circuit produces over 80 flashes per second.

21. PHYS 221 - 005 QUIZ 6 16 November 2006

The Dees of a cyclotron of radius 80 *cm* are operated at an oscillator frequency of 6.0 *MHz* to accelerate protons.

- What is the (uniform) magnetic field in the cyclotron?
- What is the speed of the protons that leave the cyclotron, in meters per second?
- What is the kinetic energy of the protons that leave the cyclotron, in electron Volts?

Note:

- $e_0 = 1.6 \times 10^{-19}\,C$
- $m_p = 1.67 \times 10^{-27}\,kg$
- $m_p c^2 = 938\,MeV$

<u>Solution</u>:

The force produced by the magnetic field must equal the mass of the proton multiplied by its (centripetal) acceleration:

$$\begin{aligned} \frac{mv^2}{r} &= qvB \\ \omega = \frac{v}{r} &= \frac{qB}{m} = 2\pi f \end{aligned}$$

Thus

$$B = \frac{m(2\pi f)}{q} = \frac{(1.67 \times 10^{-27}) \cdot (2\pi) \cdot (6 \times 10^6))}{1.6 \times 10^{-19}} = 0.39\ T$$

and

$$v = \omega R = 2\pi(6 \times 10^6) \cdot (0.8) = 3.0 \times 10^7\ m/s$$

The kinetic energy is

$$\begin{aligned} K &= \frac{1}{2}mv^2 = \frac{1}{2}(1.67 \times 10^{-27}) \cdot (3.0 \times 10^7)^2 = 0.76 \times 10^{-12}\ J \\ &= \frac{7.6 \times 10^{-13}}{1.6 \times 10^{-19}}\ eV = 4.7\ MeV \end{aligned}$$

The speed is about ten percent of the speed of light, and the non-relativistic approximation is fairly good.

22. PHYS 221 - 006 QUIZ 6 17 November 2006

The current density inside a long, solid, cylindrical wire of radius R lies in the direction of the central axis. Its magnitude varies linearly with the distance r from that central axis, $J = J_0 r/R$.

- Calculate the total current I flowing in the wire, expressed in terms of J_0 and R.
- Find the magnitude and direction of the magnetic field everywhere inside the wire.

Solution:

The total current passing through a circle of radius r that is concentric with the axis of the wire may be computed by integration of the current density:

$$I_{enc}(r) = \int J dA = \int_0^r J_0 \frac{r}{R} 2\pi r dr = \frac{2\pi J_0}{R} \frac{r^3}{3}$$

Thus the total current inside the wire is $I_{enc}(R) = 2\pi J_0 R^2/3$.

The magnetic field inside the wire is given by Ampére's Law:

$$\begin{aligned}
\oint \vec{B} \cdot \vec{d\ell} &= \mu_0 I_{enc} \\
B_t \cdot (2\pi r) &= \frac{2\pi \mu_0 J_0}{R} \frac{r^3}{3} \\
B_t &= \frac{\mu_0 J_0 r^2}{3R}
\end{aligned}$$

23. PHYS 221 - 004 QUIZ 6 28 November 2006

A solenoid having an inductance of $L = 2\ mH$ is connected in series with a $R = 40\ k\Omega$ resistor and an $\mathcal{E} = 30$ Volt (DC) battery. The switch is closed at time $t = 0$.

- What is the final current flowing in the circuit a long time after the switch is closed? How much energy is stored in the inductor at that time?
- At what time after the switch is closed is the current equal to 95 percent of its final value?

Solution:

According to Kirchhoff's Loop Equation,

$$\mathcal{E} = L\frac{dI}{dt} + RI$$

At very long times, the current is no longer changing, so that $\mathcal{E} = rI$, so that

$$I_\infty = \frac{\mathcal{E}}{R} = 304 \times 10^4 = 7.5 \times 10^{-4}\ A$$

The energy stored in the inductor is

$$U = \frac{1}{2} L I_\infty^2 = \frac{1}{2} \cdot 2 \times 10^{-3} \cdot (7.5 \times 10^{-4})^2 = 5.6 \times 10^{-10}\ J$$

From Kirchhoff's Loop Equation, and the requirement that at $t = 0$ the current vanishes, we obtain

$$\begin{aligned}
\frac{\mathcal{E}}{L} &= \frac{dI}{dt} + \frac{R}{L} I \\
\frac{\mathcal{E}}{R}\frac{d}{dt}\left[e^{Rt/L}\right] &= \frac{d}{dt}\left[I\, e^{Rt/L}\right] \\
\frac{\mathcal{E}}{R}\left[e^{Rt/L} - 1\right] &= I(t)e^{Rt/L} - I(0) \\
I(t) &= \frac{\mathcal{E}}{R}\left[1 - e^{-Rt/L}\right]
\end{aligned}$$

When the current equals $0.95 I_\infty$, we have

$$\begin{aligned}
0.95 &= 1 - e^{-Rt/L} \\
e^{-Rt/L} &= 0.05 \\
\frac{Rt}{L} &= 3.0 \\
t &= \frac{3L}{R} = \frac{3 \cdot (2 \times 10^{-3})}{4 \times 10^{-4}} = 1.5 \times 10^{-7}\, sec
\end{aligned}$$

24. PHYS 221 - 003 QUIZ 6 30 November 2006

An oscillating LC circuit consists of a $C = 3.0\, mf$ (microFarad) capacitor and an $L = 20\, mH$ (milliHenry) inductor. The maximum voltage across the capacitor is $V_m = 40$ Volts.

- What is the maximum charge on the capacitor?
- What is the maximum current through the inductor?
- What is the frequency of oscillation of the charge on the capacitor?

Solution:

The maximum charge on the capacitor is expressed in terms of the capacitance C and the maximum voltage V_m as

$$Q_m = CV_m = (3 \times 10^{-6}) \cdot 40 = 1.2 \times 10^{-4}\, C$$

The maximum energy in the capacitor $Q_m^2/(2C)$, is equal to the maximum energy in the inductor $LI_m^2/2$:

$$\begin{aligned}
\frac{1}{2}\frac{Q_m^2}{C} &= \frac{1}{2}LI_m^2 \\
I_m^2 &= Q_m^2\frac{1}{LC} = \frac{1.44\times 10^{-8}}{6\times 10^{-8}} = 0.24 \\
I_m &= 0.49A
\end{aligned}$$

The resonant angular frequency is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{6\times 10^{-8}}} = \frac{1}{2.45\times 10^{-4}} = 4100\, rad/sec$$

Then $f = \omega/(2\pi) = 600\, Hz$

Fall 2006 Examinations:
PHYS 221 - 003/004, TEST 1, 27 September 2006

1. [20 points] A ideal massless spring with spring constant $k = 20\, N/m$ hangs vertically from the ceiling. A body of mass $m = 0.5\, kg$ is attached to its free end and then released. Assume that the spring was not stretched at all before the release.

   - How far below its initial position $x = 0$ does the body descend before turning around?
   - What is the amplitude and period of the resulting simple harmonic motion?

   Solution:

   The mechanical energy of this system at a displacement x below the starting point, when the body has speed v, is

$$\frac{1}{2}kx^2 - mgx + \frac{1}{2}mv^2 = 0$$

This quantity vanishes, because body is released with speed $v = 0$ at position $x = 0$. The body stops at position

$$x = \frac{2mg}{k} = \frac{2 \cdot 0.5 \cdot 9.8}{20} = 0.49\ m$$

The block executes SHM about the position $x = mg/k = 0.245\ m$, with amplitude $A = 0.245\ m$. This can be seen by writing the relation of energy conservation in the form

$$\frac{1}{2}k\left(x - \frac{mg}{k}\right)^2 + \frac{1}{2}mv^2 = \frac{1}{2}k\left(\frac{mg}{k}\right)^2$$

The period of this motion is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.5}{20}} = \frac{2\pi}{\sqrt{40}} \approx 1 sec$$

Note that $x = mg/k$ is the equilibrium position of the body when suspended by the spring.

2. [20 points] An organ pipe produces sound in two normal modes corresponding to adjacent resonant frequencies of exactly 500 Hz and 600 Hz, respectively. The velocity of sound in air is 350 meters per second.

   - Are both ends of the pipe open, or is one end closed?
   - What is the lowest resonant frequency (fundamental mode) of this organ pipe?
   - How long is the pipe?

   Solution:

   Let us first consider the case in which both ends of the pipe are open. The ends of the pipes must be anti-nodes in that case, so that precisely n half-wavelengths of sound lie inside the pipe: $L = n\lambda/2$, or $\lambda = 2L/n$. The resonant frequencies are

$$f_n = \frac{c}{\lambda} = \frac{nc}{2L}$$

Thus, adjacent frequencies f_n and f_{n+1} satisfy the relation

$$\begin{aligned}
\frac{f_{n+1}}{f_n} &= \frac{n+1}{n} \\
\frac{600}{500} &= 1+\frac{1}{n} \\
1+0.2 &= 1+\frac{1}{n} \\
n &= 5
\end{aligned}$$

In this case, the fundamental frequency is $f_1 = 100\ Hz$, and the length of the pipe L satisfies the relation

$$\frac{c}{2L} = \frac{350}{2L} = f_1 = 100$$

Consequently $L = 1.75\ m$.

For the case of open and closed ends, the closed end is a node and the open end is an anti-node, so that $L = (2n+1)\lambda/4$. Consequently,

$$f_n = \frac{(2n+1)c}{4L}$$

and

$$\begin{aligned}
\frac{f_{n+1}}{f_n} &= \frac{2n+3}{2n+1} \\
\frac{600}{500} &= 1+\frac{2}{2n+1} \\
1+0.2 &= 1+\frac{2}{2n+1} \\
n &= 4.5
\end{aligned}$$

This case is impossible, since n is not an integer, as required.

3. [20 points] A uniform string of mass 30 grams and length 2.0 meters with fixed ends is driven in its fundamental mode, so that the amplitude of motion of the center of the string is $0.5\ cm$. The tension in the spring is 80 Newtons.

   Note: Neglect gravity.

   - Determine the resonant frequency of this fundamental mode.

- Determine the maximum velocity of a point at the center of the string.
- Determine the maximum acceleration of a point at the center of the string.

Solution:

The wavelength of vibrations of the string in its fundamental mode is $\lambda = 2L = 4.0\ m$. Thus the mass per unit length of the string is $\mu = m/L = 0.03/2.0\ kg/m = 0.015\ kg/m$. The velocity of transverse vibrations is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{800}.015 \approx 72\ m/s$$

The resonant frequency is thus

$$f = \frac{v}{\lambda} = \frac{72}{4} = 18\ Hz$$

The corresponding angular frequency is $\omega = 2\pi f \approx 110\ rad/sec$. The displacement of the string at position x; $(0 < x < L)$ and time t is

$$y(x,t) = A \sin\frac{\pi x}{L} \cos(\omega t)$$

At $x = L/2$, we obtain

$$y(t) = A \sin\frac{\pi}{2} \cos(\omega t) = A \cos(\omega t)$$

Thus

$$\begin{aligned} v_y = \frac{dy}{dt} &= -\omega A \sin(\omega t) \\ a_y = \frac{dv_y}{dt} &= -\omega^2 A \cos(\omega t) \end{aligned}$$

The maximum velocity is $v_m = \omega A = (110) \cdot (0.005) = 0.55 m/s$, whereas the maximum acceleration is $a_m = \omega^2 A = (0.55) \cdot 110 = 65 m/sec^2$. This maximum accelation at the center is more than 6 g's.

4. [20 points] Two tiny conducting balls of identical mass ($m = 1$ gram) and identical charge ($Q = 1$ microCoulomb = $10^{-6} C$) hang from non-conducting threads of length L, which are attached to a point on the ceiling. The threads each make an angle of $\theta = 30^o$ with the vertical axis.

- Show all the forces acting on the balls, including their weight.
- Determine the length of the thread, in meters.

Solution:

The point on the ceiling and the two balls lie at vertices of an equilateral triangle, so that the distance between the balls is L. Consequently, the force of Coulombic repulsion between the ball is $F = kQ^2/L^2$. The three forces acting on the ball – its weight mg (downward), the tension in the string (30^o from the vertical, and the Coulombic repulsion (horizontal) – must sum to zero. Consequently, these vectors lie along a $30^o - 60^o - 90^o$ triangle, so that

$$\begin{aligned} \tan 30^o = \frac{1}{\sqrt{3}} &= \frac{F}{mg} \\ F = \frac{kQ^2}{L^2} &= \frac{mg}{\sqrt{3}} = \frac{0.001 \cdot 9.8}{0.577} = 0.0057\,N \\ L^2 &= \frac{kQ^2}{mg/\sqrt{3}} = \frac{9 \times 10^{-3}}{0.0057} = 1.5 \\ L &= 1.2\,m \end{aligned}$$

This is a point of stable equilibrium.

5. [20 points] A total charge Q is distributed uniformly around a thin circular ring of radius R.

   - Determine the magnitude and direction of the electric field along the axis of symmetry of the ring, a distance z from the center of the ring.
   - Determine the distance at which the magnitude of the electric field is a maximum. You may express the answer in terms of R.

   Solution:

   By symmetry, the electric field lies along the axis of symmetry of the ring. The contribution of an infinitesimal piece of charge on the ring to that field component is

$$dE_z = \frac{kdq}{r^2}\cos\theta$$

where $r^2 = R^2 + z^2$ and $\cos\theta = z/r$. The total electric field may be obtained by integration over the charge distribution of the ring:

$$E_z = \int dE_z = \frac{kz}{(R^2+z^2)^{3/2}} \int dq = \frac{kQz}{(R^2+z^2)^{3/2}}$$

To obtain the value of z corresponding to the maximum field, we calculate dE_z/dz and set it to zero:

$$\frac{dE_z}{dz} = \frac{kQ}{(z^2+R^2)^{5/2}} \left((z^2+R^2) - 3z^2\right) = 0$$

We get $R^2 - 2z^2 = 0$, or $z = R/\sqrt{2}$.

6. [Extra Credit; 10 points] A particular 500 Hz isotropic sound source is barely audible (at the threshold of hearing) at a distance of 2 kilometers.

   Note: Please ignore background noise, sound absorption, sound reflection, etc.

   - How much acoustic power is being produced at the source?
   - At what distance from the source would the intensity be at the 50 dB (deciBel) level, corresponding to normal conversation.

   Solution:

   The intensity at the threshold of hearing is $I_{th} = 10^{-12}\, W/m^2$. At a distance $r = 2.0 \times 10^3 m$, the acoustic power is given by

$$P = I_{th} \times (4\pi r^2) = 4\pi(2.0\times10^3)^2 10^{-12} = 16\pi \times 10^{-6}\, W = 5\times10^{-5}\, W$$

   At $\beta = 50\, dB$, the intensity satisfies the relation

$$\begin{aligned} \beta &= 50 = 10\log_{10}\left(\frac{I}{I_{th}}\right) \\ 10^5 &= \frac{I}{I_{th}} \\ I &= 10^{-7}\, W/m^2 = \frac{P}{4\pi r^2} \end{aligned}$$

Thus

$$r^2 = \frac{P}{4\pi I} = \frac{5 \times 10^{-5}}{4\pi \times 10^{-7}} = 40$$

and $r = 6.5\ m$.

PHYS 221 - 005/006, TEST 1, 27 September 2006

1. [20 points] A 200 gram stone is attached to the bottom of an ideal massless spring that is suspended from the ceiling. The stone is pulled downward from its equilibrium position, and then released (vertical motion). If the maximum speed of the stone is 50 cm/sec, and the period is 1.0 sec, find the following:

   - The spring constant k.
   - The amplitude of motion (relative to the equilibrium position).
   - The frequency of oscillation.

   Solution:

   The frequency of oscillation of the stone is $f = 1/T = 1\ Hz$. The angular frequency is $\omega = 2\pi f = 6.3\ rad/sec$. The corresponding spring constant is $k = m\omega^2 = 0.2 \cdot (6.3)^2 = 8\ N/m$ The maximum speed is $v_m = \omega A$, so that the corresponding amplitude is $A = 0.5/(6.3) = 0.08\ m$.

2. [20 points] Two uniform strings, each of length 80 cm, are held at the same tension of 100 Newtons. The first string has a mass 10.0 grams, whereas the second has mass 10.5 grams. Each of the strings is driven in its fundamental mode.

   - Determine the frequency of vibration of the first string.
   - Is the vibrational frequency of the second string higher or lower than for the first spring? Explain.
   - Determine the frequency of beats for the two strings.

   Solution:

   The wavelength of the fundamental mode of the first string is $\lambda = 2L = 1.6\ m$. The mass per unit length of that string is $\mu = m/L = 0.01/0.8 = 0.012\ kg/m$. The velocity of transverse vibrations of that string is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{100 \cdot 80} = 89\, m/s$$

The corresponding frequency of vibration is $f = v/\lambda = 56\, Hz$.

The frequency of a vibrating spring in the fundamental mode is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

As a consequence for the two strings the ratio of frequencies is

$$\frac{f}{f'} = \sqrt{\frac{\mu'}{\mu}} = \sqrt{\frac{10.5}{10}} = 1.025$$

Thus, the frequency f' for the second spring is less than f for the first spring. We have

$$f - f' = .025 f = 1.4\, Hz$$

3. [20 points] A violin string of mass 1.0 grams and length 20 *cm* produces sound at a fundamental frequency of 500 *Hz*.

   - Determine the tension in the string.
   - Determine the wavelength and frequency of transverse waves on the string.
   - Determine the wavelength of the sound produced in the air.

   Note: The velocity of sound in the air is 350 meters per second.

   Solution:

   The mass per unit length of the violin string is $\mu = m/L = 0.001/0.2 = 0.005\, kg/m$. The wavelength for the fundamental mode is $\lambda = 2L = 0.4\, m$. The velocity of transverse vibrations is

$$v = \lambda f = 200\ m/s = \sqrt{\frac{T}{\mu}}$$

The tension is thus given by

$$T = (0.005) \cdot (200)^2 = 200N$$

The frequency of sound in air is also 500 H. The wavelength of sound is given by

$$\lambda' = v_{sound}/f = 350/500 = 0.7\ m$$

4. [20 points] Four charges, $+Q$, $+2Q$, $-Q$, and $-2Q$, are placed at vertices of a square of side a, at these respective locations: $(0,0),(a,0),(a,a),(0,a)$. Determine the magnitude and direction of the net electrostatic force acting on the charge $+Q$, which is located at $(0,0)$.

   Solution:

   The forces caused by the other three charge are tabulated below:

charge	*force*	*x − comp*	*y − comp*
$+2Q$	$\frac{2kQ^2}{a^2}$	$-\frac{2kQ^2}{a^2}$	0
$-Q$	$\frac{kQ^2}{2a^2}$	$\frac{kQ^2}{2\sqrt{2}a^2}$	$\frac{kQ^2}{2\sqrt{2}a^2}$
$-2Q$	$\frac{2kQ^2}{a^2}$	0	$\frac{2kQ^2}{a^2}$
	total	$\frac{kQ^2}{a^2}(\frac{1}{2\sqrt{2}} - 2)$	$\frac{kQ^2}{a^2}(\frac{1}{2\sqrt{2}} + 2)$

   Thus,

$$\vec{F} = \frac{kQ}{a^2}\left[-1.65\hat{\imath} + 2.35\hat{\jmath}\right]$$

   The magnitude of the force is $2.87kQ/a^2$, and its direction is 124^o from the $+x$-axis.

5. [20 points] An insulating wire of length L has a total charge Q deposited uniformly along its length. Determine magnitude and direction of the electric field at a point along the (extended) line of the wire, a distance x beyond the end of the wire.

   Solution:

   Let the wire lie in a horizontal line, $0 \le x \le L$, and let the point P be a distance y to the right if that line. The electric field points to the left. Its magnitude is given by

$$E_y = \int \frac{kdq}{r^2}$$

   The charge dq in a piece of the wire oflength dx is given by $dq = Qdx/L$. If that charge lies at location x, then its distance to the point P is $r = x + y$. Thus

$$\begin{aligned}
E_y &= \int_0^L \frac{kQ}{L} \frac{dx}{(x+y)^2} \\
&= \frac{kQ}{L} \left[-\frac{1}{x+y} \right]_{x=0}^{x=L} \\
&= \frac{kQ}{L} \left[-\frac{1}{L+y} + \frac{1}{y} \right] \\
&= \frac{kQ}{L} \frac{L}{y(y+L)} = \frac{kQ}{y(y+L)}
\end{aligned}$$

6. [Extra Credit; 10 points] One can determine the speed of blood flowing in an artery in the body by measuring the frequency shift of reflected high frequency ultrasound. Sound is produced at a frequency of 5.0 MHz, and reflected signal in blood has its frequency increased by about 5.0 kHz. Determine the speed of flow of blood in that artery.

   Note: Assume that the blood is flowing directly toward the sound source, and that the velocity of sound in blood is about 1500 meters per second.

   Solution:

   The incident wave has frequency $f_0 = 5.0\ MHz$. The frequency of sound reflected off the blood is $f_1 = f_0(1 + v/c)$, where c is the speed of sound in blood. The speed of sound detected back at the source is

$$f_2 = f_1/(1-v/c) \approx f_1(1+v/c) \approx f_0(1+2v/c)$$

The fractional change of sound is

$$\Delta f/f = (f_2 - f_0)/f_0 \approx 2v/c$$

The measured fractional change in frequency is $\Delta f/f = 5/5000 = 0.001$. Thus $v = c/2000 = 0.75\ m/s$.

PHYS 221 - 003/004, Test 2, 01 November 2006

1. [20 points] A positive charge of $+3.00\ nC$ ($1nC = 10^{-9}\ C$) is spread uniformly throughout the volume of a sphere of radius $R = 0.5$ meters. What is the magnitude and direction of the electric field at the following distances r from the center of the sphere?

   - $r = 2.0$ meters
   - $r = 0.5$ meters
   - $r = 0.1$ meters

   Solution:

   The electrical field is radial, because of spherical symmetry. Take the Gaussian surface to be a concentric sphere of radius r. According to Gauss's Law

$$\begin{aligned} \varepsilon_0 \oint \vec{E} \cdot \vec{dS} &= Q_{enc} \\ \varepsilon_0 E_r (4\pi r^2) &= Q_{enc} \\ E_r &= \frac{Q_{enc}}{4\pi\varepsilon_0} \frac{1}{r^2} \\ E_r &= \frac{kQ_{enc}}{r^2} \end{aligned}$$

   Thus, at $r = 2.0\ m$,

$$E_r = (9 \times 10^9)\,(3 \times 10^{-9})/2^2 = 6.75\,N/C$$

In addition, at $r = 0.5\,m$, $E_r = 108\,N/C$.

For $r \geq R$, $Q_{enc} = Q$ and $E_r = kQ/r^2$. Inside the sphere for $r < R$, $Q_{enc} = Q(r/R)^3$ and

$$E_r = \frac{kQ}{r^2}\frac{r^3}{R^3} = \frac{kQr}{R^3}$$

Thus, for $r = 0.1\,m$,

$$E_r = (9 \times 10^9)\,(3 \times 10^{-9})\,(0.1)/(0.5)^3 = 22\,N/C$$

2. [20 points] An insulating sphere of radius $R = 0.1$ meters contains a charge that is uniformly spread throughout its interior. It observed that there is a net electric flux of $8 \times 10^{-4}\ Nm^2/C$ passing out of a concentric spherical Gaussian surface of radius $r = 0.5$ meters.

   - What is the electric potential at a distance of 1.0 meters from the center of the sphere, in Volts?
   - What is the total charge on the sphere, in Coulombs?

   Solution:

   Gauss's Law has the form $\varepsilon_0 \Phi_E = Q_{enc}$, so that the charge inside a concentric spherical Gaussian surface of radius $0.5\,m$ is $Q_{enc} = (8.85 \times 10^{-12}) \cdot (8 \times 10^{-4}) = 7 \times 10^{-15}\,C$. This is the charge inside the sphere. The electrostatic potential at an arbitrary point outside the sphere, at a distance r from its center, is $V = kQ_{enc}/r = (9 \times 10^9) \cdot (7 \times 10^{-15})/1.0 = 6.4 \times 10^{-5}\,V$.

3. [20 points] A capacitor of unknown capacitance C is charged to 200 Volts, and the charging source is then disconnected. Then it is connected across an uncharged 50 μF capacitor. The final potential difference across the 50 μF capacitor is 40 Volts. Note: $1\,\mu F = 10^{-6}\,F$.

   - What is the unknown capacitance C?

- How much charge is on each capacitor at the conclusion?
- How much electrical energy is stored in the system, before and after the connection? Does it change? Why or why not?

<u>Solution</u>:

Let q_0 be the charge initially on the capacitor C; note that $q_0 = C(200)$. That charge is distributed among the two capacitors after the connection: $q_0 = q + q'$. Note that $q' = (50 \times 10^{-6})\,(40) = 2.0\,mC$, and $q = 40C$. Thus

$$\begin{aligned} q_0 &= q + q' \\ 200C &= 40C + 2 \times 10^{-3} \\ C &= 12.5\,\mu F \\ q_0 &= 200 \cdot (12.5 \times 10^{-6}) = 2.5\,mC \end{aligned}$$

The energy initially stored in the capacitor is $1/2(12.5 \times 10^{-6})(200)^2 = 250\,mJ$. The energy stored afterward is $1/2(62.5 \times 10^{-6})(40)^2 = 50\,mJ$. Note that $4/5$ of the energy has been dissipated in the process.

4. [20 points] A steel trolley car rail has a cross-sectional area of 0.01 square meters. The electrical resistivity of steel is $3.00 \times 10^{-7}\ \Omega m$.

   - What is the resistance of 10 km of trolley rail?
   - If a current of 100 A is passing through the trolley rail from end to end, how much energy per unit time is dissipated within the rail?.

   <u>Solution</u>:

   The resistance of the rail is

$$R = \frac{\rho L}{A} = \frac{3 \times 10^{-7} \cdot 10^4}{10^{-2}} = 0.3\,\Omega$$

   The power dissipated is

$$P = I^2 R = (100)^2\,(0.3) = 3000\,W$$

5. [20 points] A capacitor of capacitance 2.0 μF "leaks" slightly, in that charge passes from one plate to the other over the course of time. The charge on one of the plates drops to half its value in 20 minutes.

Note 1 $\mu F = 10^{-6}\, F$.

What is the equivalent resistance between the capacitor plates?

Solution:

The charge on the capacitor at time t is

$$q(t) = q_0 e^{-t/\tau}$$

Thus

$$\begin{aligned} q_0/2 &= q_0 e^{-\tau} \\ 2 &= e^{t/\tau} \\ \ln 2 = 0.69 &= \frac{t}{\tau} \\ \tau &= 20\, min/0.69 = 28.85\, min = 1730\, sec \end{aligned}$$

The resistance R is

$$R = \frac{\tau}{C} = \frac{1730}{2 \times 10^{-6}} = 8.7 \times 10^{8}\, \Omega$$

6. [Extra credit; 10 points] A resistor of unknown resistance is connected between the terminals of an ideal 9.0 Volt battery. The rate of dissipation of energy in the resistor is 0.05 Watts.

- Determine the resistance of the resistor.
- If the same resistor is placed between the terminals of an ideal 36 Volt battery, determine the rate of dissipation of energy in the resistor, in Watts.

Solution:

From the formula for power lost in the resistor $P = V^2/R$, it follows that the resistance is

$$R = \frac{V^2}{P} = \frac{9^2}{0.05} = 1600\,\Omega$$

The power lost in the $36\,V$ battery is

$$P' = \frac{36^2}{1600} = 0.80\,W$$

PHYS 221- 005/006, TEST 2, 01 November 2006

1. [20 points] Charge of uniform volume density $\rho = +2.0\,nC/m^3$ fills a large (infinite) slab between $x = -1.0$ meters and $x = +1.0$ meters. There are no other charges in the vicinity of this large slab. What is the magnitude and direction of the electric field at any point with the following coordinates:

   Note: $1nC = 10^{-9}\,C$.

   - $x = -2.0$ meters
   - $x = -0.2$ meters
   - $x = +0.0$ meters
   - $x = +0.6$ meters
   - $x = +4.0$ meters

   Solution:

   The charge distribution is symmetric about the plane $x = 0$, which lies at the center of the slab. Thus, the electric field is zero along that slab, at $x = 0$. The electric field outside lies in the direction of increasing x for $x > 0$, whereas for $x < 0$ it lies in the direction of decreasing x. The charge per nit area in the slab is $\sigma = \rho t$, where $T = 2\,m$ is the thickness of the slab. Thus, $\sigma = 2 \times 10^{-9} \cdot 2 = 4 \times 10^{-9}\,C/m^2$. It follows from Gauss's law for a pillbox of area A parallel to the slab and enclosing a piece of it that that the magnitude of the electric field E outside the slab is given by

$$\begin{aligned}
\varepsilon_0 \oint \vec{E} \cdot \vec{dS} &= Q_{enc} \\
\varepsilon_0 E(2A) &= \sigma A \\
E &= \frac{\sigma}{2\varepsilon_0} = \frac{4 \times 10^{-9}}{2 \cdot 8.85 \times 10^{-12}} = 230\,N/C
\end{aligned}$$

To determine the field inside the slab at a distance x from its center, we take a Gaussian pillbox of area A with bases at on its center plane and at a distance x from the center. Applying Gauss's law, we get

$$\begin{aligned} \varepsilon_0 \oint \vec{E} \cdot d\vec{W} &= Q_{enc} \\ \varepsilon_0 E(x)(A) &= \rho x \\ E &= \frac{\rho x}{\varepsilon_0} = \frac{2 \times 10^{-9} x}{\cdot 8.85 \times 10^{-12}} = 230\, x\, N/C \end{aligned}$$

Thus, at $x = -0.2$, $E_x = -45\ N/C$, whereas at $x = 0.6$, $E_x = +140\ N/C$.

2. [20 points] A charge $+Q$ is distributed uniformly throughout a spherical volume of radius R. Let the electrostatic potential V be zero at infinity.

- What is the electrostatic potential inside the sphere: $r < R$?
- What is the electrostatic potential outside the sphere: $r > R$?
- What is the potential difference between the center of the sphere ($r = 0$) and its surface ($r = R$)?

Solution:

The electric field is determined through Gauss's Law with a concentric spherical Gaussian surface. Outside the sphere ($r > R$) we obtain

$$\begin{aligned} \varepsilon_0 \oint \vec{E} \cdot d\vec{S} &= Q_{enc} \\ \varepsilon_0 E_r (4\pi r^2) &= Q \\ E_r = -\frac{dV}{dr} &= \frac{kQ}{r^2} \\ V(r) &= \frac{kQ}{r} \end{aligned}$$

Inside the sphere, $Q_{enc} = Qr^3/R^3$, so that

$$\begin{aligned} E_r = -\frac{dV}{dr} &= \frac{kQ}{r^2}\frac{r^3}{R^3} = \frac{kQr}{R^3} \\ V(r) &= -\frac{kQr^2}{2R^3} + V_0 \end{aligned}$$

The constant of integration is determined by the requirement that, at $r = R$, $V(R) = kQ/R$, as the point is approached from both inside and outside the sphere. Thus, $V_0 = 3KQ/(2R)$. Finally,

$$V(0) - V(R) = \frac{3kQ}{2R} - \frac{kQ}{R} = \frac{kQ}{2R}$$

3. [20 points] A dielectric sphere capacitor is placed by putting dielectric material of dielectric constant $\kappa = 50$ between concentric spherical metallic plates. The inner plate has a radius of $r_1 = 1.00\ mm$, and the outer plate has a radius of $r_2 = 1.05\ mm$.

   Note: $1\ mm = 10^{-3}\ m$.

   - Determine the capacitance, in Farads.
   - With a charge of 20 nC on the capacitor, how much electrical energy is stored in it?

   Solution:

   Let us apply Gauss's law to a concentric spherical surface lying entirely within the dielectric:

$$\begin{aligned} \varepsilon_0 \oint \kappa \vec{E} \cdot \vec{dS} &= Q_{enc} \\ \varepsilon_0 E_r \cdot 4pir^2 &= -Q \\ E_r &= -\frac{kQ}{\kappa r^2} \end{aligned}$$

   The potential difference between the plates of the capacitor are

$$\Delta V = -\int_a^b E_r dr = \left[-\frac{kQ}{\kappa r}\right]_a^b = \frac{kQ}{\kappa}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{kQ}{\kappa}\frac{b-a}{ab} = Q/C$$

The capacitance is

$$C = \frac{\kappa}{k}\frac{ab}{b-a} = \frac{50}{9\times10^9}\frac{1.05\times16{-}6}{5\times10^{-5}} = 1.16\times10^{-10}\,F$$

The stored electrical energy is

$$U = \frac{Q^2}{2C} = \frac{(2\times10^{-8})^2}{2\cdot1.16\times10^{-10}} = 1.7\times10^{-6}\,J$$

4. [20 points] A fuse in an electric circuit is a wire that is designed to melt, and thereby to open the circuit, if the current exceeds a predetermined value. Suppose that the material to be used melts when the current density rises to $500\,A/cm^2$.

   - What diameter of cylindrical wire should be used to make a fuse that will limit the current to $1.0\,A$?
   - What must be the resistivity ρ of the material in the wire, if there is an electric field of $10^2\,V/m$ in the wire when it melts?

   Solution:

   The current density in the wire is $J = 5\times10^6 A/m^2$, so that for an electric field $e = 100V/m$ in the wire, the resistivity must be $\rho = E/J = 100/(5\times10^6) = 2\times10^{-5}\,\Omega m$.

   When a current $I = 1\,A$ is flowing, the area of the wire is

$$A = 1/(5\times10^6) = 2\times10^{-7}\,m^2$$

   Setting $A = \pi D^2/4$, we obtain a diameter $D = 0.50\,mm$.

5. [20 points] Two ideal batteries, each of electromotive potential $\mathcal{E} = +12$ Volts, are each connected in series with a $6\,\Omega$ resistors and then in parallel with one other, with opposite polarities. This combination is then placed across a third resistor, which also has a resistance of $6\,\Omega$. Under steady-state conditions, determine the magnitude and direction of the current through each resistor, the power loss in each resistor, and the power provided by each battery.

Solution:

The current passing through the third resistor is zero, since the batteries have opposite polarities. Furthermore, the potential drop across each battery-resistor combination is zero. Thus, The current flowing through each resistor is $I = 2\,A$. Each battery provides power $P = IV = 24\,W$, and the resistors each convert $24\,W$ into thermal energy.

6. [Extra Credit; 10 points] In Millikan's experiment, and oil drop of radius $2.0\ \mu m$ and density $0.9\ g/cm^3$ is suspended (at rest) in a chamber, with a downward electric field of $2.0 \times 10^5\ N/C$ is applied. Find the charge on the drop, in Coulombs. Do not neglect gravity.

   Note: ($1\mu m = 10^{-6}$ meters)

   Solution:

   For balance of the forces we obtain

$$
\begin{aligned}
qE &= -mg \\
qE &= -\rho \frac{4\pi r^3}{3} g \\
q &= -\frac{4\pi}{3} \frac{r^3 \rho g}{E} \\
&= -\frac{4\pi}{3} \frac{(2 \times 10^{-6})^3 \cdot 900 \cdot 9.8}{2 \times 10^5} \\
&= = 1.44 \times 10^{-18}\,C
\end{aligned}
$$

   This corresponds to about nine electron charges.

PHYS 221 - 003/004, FINAL Examination, 13 December 2006

1. [25 points] The scale of a light spring balance that reads from 0 to $m = 20\,kg$ is $x = 8.0\ cm$ long. A package suspended from the balance is found to oscillate vertically from the spring balance with a frequency of $1.5\,Hz$.

   - What is the spring constant k, in N/m?
   - What is the mass of the package, in kg?

Solution:

The spring constant is $k = mg/x = 20 \cdot 9.8/0.08 = 2450\ N/m$. The angular frequency of vibration is $\omega = 2\pi f = 2\pi(1.50) = 9.42\ rad/sec$. From the relation $\omega^2 = k/m$ we determine the mass:

$$m = \frac{k}{\omega^2} = \frac{2450}{(9.42)^2} = 28\ kg$$

2. [25 points] Two charged thin concentric spherical shells have radii 10 cm and 20 cm, respectively. The charge on the inner shell is $+6\ nC$, whereas the charge on the outer shell is $+12\ nC$. Find the electric field at these distances from the center of the shells.

   - $r = 15 cm$
   - $r = 30\ cm$

   Solution:

   For a concentric spherical Gaussian surface of radius r, Gauss's Law yields

$$\begin{aligned} \varepsilon_0 \oint \vec{E} \cdot \vec{dS} &= Q_{enc} \\ \varepsilon_0 (4\pi r^2) E_r &= Q_{enc} \\ E_r &= \frac{kQ_{eenc}}{r^2} \end{aligned}$$

   For $r = 15\ cm$, $Q_{enc} = 6\ nC$, so that

$$E_r = \frac{(9 \times 10^9) \cdot (6 \times 10^{-9})}{(0.15)^2} = 2400\ N/C.$$

   For $r = 30\ cm$, $Q_{enc} = 18\ nC$, so that

$$E_r = \frac{(9 \times 10^9) \cdot (18 \times 10^{-9})}{(0.30)^2} = 1800\ N/C.$$

3. [25 points] An air-filled parallel-plate capacitor has a capacitance of $C_0 = 1.5\ pF$. The separation between plates d is doubled, and a dielectric material is inserted between the plates. The new capacitance is $C_1 = 6.0\ pF$.

   - Determine the dielectric constant κ of the material.
   - If charges of ± 3 microCoulombs (mC) remain on the plates of the capacitor throughout this process, calculate the energy stored in the capacitor (in Joules) in the beginning and in the end.

   Solution:

   The capacitance of the air-filled capacitor is $C_0 = \varepsilon_0 A/d$, whereas for the dielectric-filled capacitor we obtain $C_1 = \kappa\varepsilon_0 A/(2d)$. It follows that

   $$C_1/C_0 = \kappa/2 = 4$$

   Thus, the dielectric constant is $\kappa = 8$.

   The energy originally stored in the capacitor is

   $$U_0 = \frac{Q_0^2}{2C_0} = \frac{(3\times 10^{-6})^2}{2(1.5\times 10^{-12})} = 3\ J$$

   The energy stored at the end is

   $$U_1 = \frac{Q_0^2}{2C_1} = \frac{(3\times 10^{-6})^2}{2\cdot(6\times 10^{-12})} = 0.75\ J$$

4. [25 points] In a certain cyclotron, protons move in a circle of radius 0.5 meters. The magnitude of the magnetic field is 3.0 T, and the direction of the field is perpendicular to the orbital plane.

   - What is the oscillator frequency?
   - What is the speed of the protons, in meters per second?
   - What is the kinetic energy of the protons, in electron Volts?

Solution:

It follows from the Lorentz force relation that

$$\begin{aligned}\frac{mv^2}{r} &= qvb \\ \omega &= \frac{v}{r} = \frac{qB}{m} = \frac{(1.6\times10^{-19})\cdot 3}{1.67\times10^{-27}} = 2.87\times10^{8}\ rad/sec\end{aligned}$$

The corresponding frequency is $f = \omega/(2\pi) = 46\ MHz$. The speed of the protons at ejection is $v = \omega R = (2.87\times10^{8})\cdot(0.5) = 1.5\times10^{8}\ m/s$. The corresponding kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(1.67\times10^{-27})(1.44\times10^{8})^2 = 106\ MeV$$

Although this is nearly half the speed of light, the non-relativistic approximation is fairly good.

5. [25 points] the inductance of a closely wound coil is such that an EMF of $3.0\ mV$ is induced when the current changes at the rate of $5.0\ Amps$ per second. A steady current of $1.0\ Amps$ through the coil produces a flux of 4.0×10^{-5} Webers through each turn.

   - Calculate the inductance of the coil, in Henries.
   - How many turns N are there in the coil?

   Solution:

   From Faraday's Law,

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

   it follows that the inductance is $L = (3\times10^{-3})/5 = 6\times10^{-4}\ H$. The flux per turn is $\Phi_0 = BA = 4\times10^{-5}\ W$. Thus,

$$\begin{aligned}N\Phi_0 &= \Phi = LI \\ N\cdot(4\times10^{-5}) &= (6\times10^{-4})1 \\ N &= 15\ turns\end{aligned}$$

6. [25 points] As a parallel-plate capacitor with circular plates 30 cm in diameter is being charged, the current density of the displacement current between the plates is uniform and has a magnitude of $J_d = 20\ A/m^2$.

- Calculate the magnitude of the magnetic field B at a distance of $4 = 20cm$ from the center of symmetry of this region.
- Calculate dE/dt in this region.

<u>Solution</u>:

The displacement current passing through a concentric ring of radius r is

$$I_d = \pi r^2 J_d = \pi(0.2)^2(20) - 1.41\,A$$

According to Maxwell's extension of Ampère's Law

$$\begin{aligned} \oint \vec{B}\cdot\vec{d\ell} &= \mu_0 I_d \\ B_t(2\pi r) &= \mu_0 I_d \\ B_t &= \frac{\mu_o I_d}{2\pi r} = \frac{(4\pi\times 10^{-7})\cdot(1/41}{2\pi(-02)} = 1.41\times 10^{-6}\,T \end{aligned}$$

From the relation

$$I_d = \varepsilon_0 \int \frac{\partial\vec{E}}{\partial t}\cdot\vec{dS} = \varepsilon_0 A\frac{\partial E}{\partial t}$$

We obtain

$$J_d = \varepsilon_0\frac{\partial E}{\partial t}$$

or

$$\frac{\partial E}{\partial t} = \frac{J_d}{\varepsilon_0} = \frac{20}{8.85\times 10^{-12}} = 2.3\times 10^{-12}\,\frac{V}{ms}$$

7. [Extra credit; 10 points] You are standing at a distance D from an isotropic point source of sound. You walk 100 meters directly toward the source, and observe that the intensity of sound has doubled. Calculate the distance D.

   Solution:

   The intensity I of isotropically produced sound at a distance D from a source of acoustic power P is $I = P/(4\pi D^2)$. At distance $D - 100$ the intensity is $2I$, so that $2I = P/(4\pi(D-100)^2)$. Consequently

$$
\begin{aligned}
P = 4\pi D^2 I &= 8\pi(D-100)^2 I \\
D^2 &= 2(D-100)^2 \\
D &= \sqrt{2}(D-100) \\
D &= \frac{100}{1-1/\sqrt{2}} = 100(2+\sqrt{2}) = 340\ m
\end{aligned}
$$

PHYS 221 - 005/006, Final Examination, 14 December 2006

1. [25 points] A nylon guitar string has a linear density of $\mu = 6.0\ g/m$ and is under a tension $T = 120\ N$. The fixed supports are a distance $D = 80\ cm$ apart. The string is oscillating in a standing wave pattern consisting of three loops. Calculate the speed, wavelength, and frequency of the traveling waves whose superposition gives this standing wave.

   Solution:

   Each loop corresponds to a half-wavelength, so that $D = 3/2\lambda$, or $\lambda = 2/3 \cdot 80 = 53\ cm$. The velocity of transverse vibrations is

$$
v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{120}{0.006}} = \sqrt{200000} = 140\ m/s
$$

   The transverse displacement $y(x,t)$ is given by

$$
y = A \sin\frac{3\pi x}{D}\ \sin\omega t = \frac{A}{2}\left[\cos(kx - \omega t) - kx + \omega t)\right]
$$

   The frequency of vibration is $f = v/\lambda = 140/0.53 = 265\ Hz$.

2. [25 points] Two particles, each with charges $+12nC$, are placed at two of the vertices of an equilateral triangle with edge length $a = 0.5$ meters.

   - What is the magnitude and direction of the electric field at the third vertex of the triangle?
   - What is the electrostatic potential at that point?

   Solution:

   The electric field at the third vertex is the sum of the fields produced by the two charges: $\vec{E} = \vec{E}_1 + \vec{E}_2$. From symmetry, the field lies along the altitude to the side joining the charges, and away from the triangle. The magnitude of each of the fields is $E_1 = E_2 = kQ/a^2$. The magnitude of the total field is

$$
\begin{aligned}
E &= 2E_1 \cos 30^o = \sqrt{3}kQ/a^2 \\
&= \sqrt{3}(9 \times 10^9) \cdot (12 \times 10^{-9})/(0.5)^2 = 720\, V/m
\end{aligned}
$$

   The electric potential at the vertex is

$$
V = V_1 + V_2 = 2kQ/a = 2(9 \times 10^9) \cdot (12 \times 10^{-9})/(0.5) = 432\, V
$$

3. [25 points] An RC circuit is connected across a 20 Volt battery, and the switch is closed at time $t = 0$. The resistance is $R = 2000$ Ohms, and the capacitance is $C = 80$ microFarads.

   - What is the final charge on the capacitor, in Coulombs?
   - At what time after the switch is closed is the capacitor charged to half of its final value?

   Solution:

   The equilibrium charge on the capacitor is $Q_0 = CV = 20 \cdot 8 \times 10^{-5} = 1.6 \times 10^{-3}\, C$. The time constant for this circuit is $\tau = RC = 2 \times 10^{-3} \cdot 8 \times 10^{-5} = 0.16\, sec$. The charge on the capacitor at time is $Q(t)$:

$$
\begin{aligned}
Q(t) &= Q_0\left[1-e^{-t/\tau}\right] \\
\frac{Q_0}{2} &= Q_0\left[1-e^{-t/\tau}\right] \\
\frac{1}{2} &= e^{-t/\tau} \\
\ln 2 &= \frac{t}{\tau} \\
t &= 0.69\cdot 0.16 = 0.11\,sec
\end{aligned}
$$

4. [25 points] A long solenoid with $n = 5000$ turns per meter carries a current I. An electron moves within the solenoid in a circle of radius $R = 3.0\ cm$, perpendicular to the solenoid axis. The speed of the electron is $v = 10^6$ meters/second. Find the current in the solenoid.

   Solution:

   The Lorentz force relation $\vec{F} = m\vec{a} = q\vec{V}\times\vec{B}$ yields the relation

$$
\begin{aligned}
\frac{mv^2}{r} &= qvB \\
B &= \frac{mv}{qR} = \frac{(9.1\times 10^{-31})\cdot 10^6}{(1.6\times 10^{-19})\cdot(0.03)} = 1.9\times 10^{-4}\ T
\end{aligned}
$$

   It follows from Ampère's Law that

$$
\begin{aligned}
B &= \mu_0 n I \\
I &= \frac{B}{\mu_0 n} = \frac{1.9\times 10^{-4}}{(4\pi\times 10^{-7})\cdot(5\times 10^3)} = 0.03\,A
\end{aligned}
$$

5. [25 points] In an oscillating LC circuit, with $L = 8.0\ mH$ and $C = 2.0\ \mu F$. At time $t = 0$, the current is maximum at $20\ mA$.

   - What is the maximum charge on the capacitor during the oscillations?
   - At what earliest time $t > 0$ is the rate of change of energy in the capacitor a maximum? What is that maximum rate of change, in Joules per second?

Solution:

At time $t = 0$ the charge on the capacitor is $Q_0 = 0$, whereas the current in the inductor is $I_0 = 0.02\,A$. According to Kirchhoff's loop equation,

$$\begin{aligned} L\frac{dI}{dt} + \frac{Q}{C} &= 0 \\ \frac{d^2Q}{dt^2} + \frac{1}{LC}Q &= 0 \end{aligned}$$

The angular frequency of (harmonic) oscillation of charge is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(8\times10^{-5})(2\times10^{-8})}} = 7900\ rad/sec$$

Thus

$$\begin{aligned} Q(t) &= Q_0\ \sin\omega t \\ I(t) &= \omega Q_0\ \cos\omega t \end{aligned}$$

It follows that the maximum charge on the capacitor is

$$Q_0 = I_0\omega = 0.02/7900 = 2.6\times10^{-6}\,C$$

The energy in the capacitor at time t is $U(t)$:

$$\begin{aligned} U(t) &= \frac{Q(t)^2}{2C} = \frac{Q_0^2}{2C}\ \sin^2\omega t \\ U'(t) &= \frac{\omega Q_0^2}{C}\ \sin\omega t\cos\omega t = \frac{\omega Q_0^2}{2C}\ \sin(2\omega t) \end{aligned}$$

The maximum rate of change is

$$U'_{max} = \omega Q_0^2/(2c) = (7900)(2.5\times10^{-6})^2/(4\times10^{-6}) = 2.5\times10^{-2}\,J$$

6. [25 points] The index of refraction of Benzene is $n = 1.8$.

- If a small coin lies at a depth of $d = 15\ cm$ in a pool of Benzene, determine the apparent depth y of the coin, as seen from above.
- What is the critical angle for a light ray traveling in Benzene toward a layer of air above the Benzene.

<u>Solution</u>:

If a light ray is incident from above at a small angle θ to the vertical direction, it is refracted to an angle ϕ to the vertical:

$$\begin{aligned} \sin\theta &= n\sin\phi \\ \tan\theta &\approx n\tan\phi \\ \frac{x}{y} &\approx n\frac{x}{d} \\ y &\approx \frac{d}{n} = \frac{15}{1.8} = 8\ cm \end{aligned}$$

Note that the apparent horizontal location x of the coin is unaffected by refraction.

The critical angle ϕ in Benzene is determined by the relation

$$\begin{aligned} n\sin\phi &= \sin 90^o = 1 \\ \sin\phi &= \frac{1}{n} = \frac{1}{.8} = 0.55 \\ \phi &= 33.7^o \end{aligned}$$

7. [Extra Credit; 10 points] How much work is required to assemble four charges $Q = 5\ nC$ at the vertices of a square of side $a = 0.2$ meters? Assume that the charges are infinitely far apart before this assembly.

<u>Solution</u>:

The potential energy stored in a configuration of charges q_i is a sum of the potential energies for each pair of charges:

$$U = k \sum_{i<j} \frac{q_i q_j}{r_{ij}}$$

Four of pairs of charges are separated by a distance a, whereas the separation of other two pairs is $\sqrt{2}a$. Thus

$$\begin{aligned} U &= 4k\frac{Q^2}{a} + 2k\frac{Q^2}{\sqrt{2}a} = \frac{kQ^2}{a}(4+\sqrt{2}) \\ &= \frac{(9\times 10^9)\cdot(5\times 10^{-9})^2}{0.2}(4+\sqrt{2}) = 6.1\times 10^{-6}\ J \end{aligned}$$