Physics 403: Relativity

Takehome Final Examination

Due 07 May 2007

1. Suppose that a beam of protons (rest mass \( m = 938 \text{ MeV}/c^2 \)) of total energy \( E = 300 \text{ GeV} \) strikes a proton target at rest. Determine the largest mass of a particle \( X \) that could be produced in the following reaction:

\[ p + p \rightarrow p + p + X \]

Solution:
The proton projectile (with mass \( m \) and total energy \( E \)) is incident upon a proton at rest. Thus, the total initial energy and momentum in the laboratory frame are given as follows:

\[
E_{\text{lab}}^i = E + mc^2 = 300.938 \text{ GeV}
\]

\[
|\vec{p}_{\text{lab}}^i|^2 = E^2 - m^2c^4
\]

The final state consists of two protons (with energies \( E_1 \) and \( E_2 \), respectively) and the \( X \)-particle (with mass \( M_X \) and energy \( E_X \)). Thus, the final energy and momentum are

\[
E_{\text{lab}}^f = E_1 + E_2 + E_X
\]

\[
|\vec{p}_{\text{lab}}^f|^2 = \vec{p}_1 + \vec{p}_2 + \vec{p}_X
\]

Let us view this collision in the center of momentum frame, in which the total momentum is zero — before as well as after the collision. We make use of Lorentz invariance of the quantity \( E^2 - c^2 p^2 \) to determine the energy in the center of momentum frame:

\[
(E_{\text{cm}}^f)^2 = (E_{\text{cm}}^i)^2 - c^2 (p_{\text{cm}}^i)^2 = (E + mc^2)^2 - E^2 + m^2c^4 = 2mc^2(E + mc^2)
\]

At the threshold for production of the \( X \)-particle, each of the three final state particles will be at rest in that frame. As a consequence

\[
E_{\text{cm}}^f = 2mc^2 + m_Xc^2
\]
Setting these two energies equal, we obtain

\[ E_{\text{cm}}^i = E_{\text{cm}}^f \]
\[ \sqrt{2mc^2(E + mc^2)} = 2mc^2 + m_X c^2 \]
\[ \sqrt{2 \cdot 0.938 \cdot 300.938 \text{GeV}} = 1.876 \text{GeV} + m_X c^2 \]
\[ 23.780 = 1.876 + m_X c^2 \]
\[ m_X c^2 = 21.884 \text{GeV} \]

2. A rocket ship accelerates directly away from the earth (radius \( R \)) with a constant acceleration \( g \), as seen in the rest frame of the rocket.

- Calculate the angular size of the earth, as viewed from the rocket, expressed in terms of the proper time of flight on the rocket.
- Show that, as the proper time gets large, the angular size approaches the limiting value \( 2gR/c^2 \).

Solution:
Let us restrict considerations to linear motion of the rocket along the \( x \)-direction, with the other spatial coordinates set to zero. The space time coordinates of the rocket ship in the frame of the earth at earth time \( t \) are \( x^\mu = (ct, x) \). The four-velocity is given by

\[ u^\mu = \frac{dx^\mu}{d\tau} = \left( \frac{dt}{d\tau}, \frac{dx}{d\tau} \right) = \gamma \left( c, \frac{dx}{dt} \right) = \gamma (c, v) \]

This four-velocity always satisfies the relation

\[ u_\mu u^\mu = c^2 \]

Let us define the four-acceleration as

\[ a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = \left( c \frac{d^2t}{d\tau^2}, c \frac{d^2x}{d\tau^2} \right) \]

It follows directly from the definition that

\[ a_\mu u^\mu = 0 \]
in every inertial frame. In the instantaneous rest frame of the rocket (co-moving frame) the speed of the rocket is zero, so that $v^\mu = (c, 0)$ and $a^\mu = (0, g)$. We may calculate the frame-invariant quantity $a_\mu a^\mu = -g^2$.

Let us next calculate these two invariant quantities in the earth’s frame:

$$a_\mu u^\mu = 0 = \gamma \left( c^2 \frac{d^2 t}{d\tau^2} - v \frac{d^2 x}{d\tau^2} \right)$$

$$a_\mu a^\mu = -g^2 = \left( c \frac{d^2 t}{d\tau^2} \right)^2 - \left( \frac{d^2 x}{d\tau^2} \right)^2$$

Thus

$$c^2 \frac{d^2 t}{d\tau^2} = v \frac{d^2 x}{d\tau^2}$$

$$\left( 1 - \frac{v^2}{c^2} \right) \left( \frac{d^2 x}{d\tau^2} \right)^2 = g^2$$

$$\frac{d^2 x}{d\tau^2} = d\tau (\gamma v) = \gamma g$$

$$\gamma \frac{dv}{d\tau} \left( 1 + \gamma^2 \frac{v^2}{c^2} \right) = \gamma g$$

$$\gamma \frac{dv}{d\tau} = g$$

$$\frac{dv}{1 - v^2/c^2} = g d\tau$$

We begin with $v = 0$ at proper time $\tau = 0$, so that

$$v = c \tanh \frac{g\tau}{c}$$

$$\gamma = \cosh \frac{g\tau}{c}$$

$$\frac{dx}{d\tau} = \gamma v = c \sinh \frac{g\tau}{c}$$

Since the rocket starts at $x_0 = R_c$ at proper time $\tau = 0$, its position at proper time $\tau$ is
\[ x = \frac{c^2}{g} \left( \cosh \frac{g \tau}{c} - 1 \right) + R_e \]

According to an observer inside the rocket, the distance to earth is Lorentz-contracted:

\[ x' = \frac{x}{\gamma} = \frac{c^2 \left( \cosh \frac{g \tau}{c} - 1 \right) + g R_e}{g \cosh g \tau} \]

The angular size of the earth, as seen on the rocket, is \( \theta' \), where

\[ \tan \frac{\theta'}{2} = \frac{y'}{x'} = \frac{R}{x'} = \frac{g R_e \cosh g \tau/c}{c^2 \left( \cosh \frac{g \tau}{c} - 1 \right) + g R_e} \]

In the limit of large proper time \( \tau \) we obtain the limiting angular size:

\[ \tan \frac{\theta'}{2} \to \frac{g R_e}{c^2} \]

The rocketeer thus always has a (small) glimpse of the receding earth — at least in principle!

3. A black hole at the center of our galaxy has a radio source moving in a circular orbit about it with a visual radius \( \theta \) of about 0.2 arc seconds, and with a period \( T \) of about 30 years. The galactic center is a distance \( D \) of about 10 kiloparsecs away from us. Using Newtonian gravity, determine the mass of the black hole.

**Solution:**

We shall use nonrelativistic kinematics and Newtonian gravity, under the assumption that the orbiting radio source (mass \( m \) and orbit radius \( r \)) is much lighter than the black hole (mass \( M \)), as well as far away from it. Consequently

\[ F = \frac{GMm}{r^2} = m \omega^2 r \]

\[ \left[ \frac{2\pi}{T} \right]^2 = \omega^2 = \frac{GM}{r^2} \]

\[ T^2 = \frac{4\pi^2 r^3}{GM} \]

This formula would also apply for the orbit of the earth about the sun (mass \( M_S \)), with its (average) radius \( r_E \) of 1 AU \( \approx 1.5 \times 10^{11} \) m and period \( T_E = 1 \) year. That is
\[ T_E^2 = \frac{4\pi^2 r_E^3}{GM_S} \]

Taking the ratio of these two expressions, we obtain

\[
\left( \frac{T_E}{T} \right)^2 = \left[ \frac{r_E}{r} \right]^3 \frac{M}{M_S}
\]

\[ M = M_S \left( \frac{T_E}{T} \right)^2 \left[ \frac{r}{r_E} \right]^3 \]

We may thus determine \( M \) (in solar masses) in terms of \( T \) in years and \( r \) in AU by using the formula

\[ M = T^2 r^3 \]

The time \( T \) is 30 years. What is the distance \( r \)?

First we express the distance \( D \) to the galactic center in AU:

\[ D = 10^4 \text{ parsec} \times \frac{1 \text{ AU}}{5 \times 10^{-6} \text{ parsec}} = 2 \times 10^9 \text{ AU} \]

Next, we obtain the visual radius \( \theta \) in radians:

\[ \theta = 0.2 \text{ arcsec} \times \frac{1 \text{ deg}}{3600 \text{ arcsec}} \times \frac{\pi \text{ rad}}{1 \text{ deg}} = 9.7 \times 10^{-7} \text{ radians} \]

Thus the orbit radius is \( r = D\theta = 1900 \text{ AU} \) and \( M = 1900^3/30^2 = 8 \times 10^6 \) solar masses. Since the solar mass is \( 2 \times 10^{30} \text{ kg} \), we may express the mass as \( M = 1.6 \times 10^{37} \text{ kg} \).

The Schwarzschild radius of the black hole may then be computed

\[ 2R_0 = \frac{2GM}{c^2} = 2.4 \times 10^{10} \text{ m} \]

Since \( 1 \text{ AU} = 1.5 \times 10^{11} \text{ m} \), we obtain

\[ r = 1900 \text{ AU} = 3 \times 10^{14} \text{ m} \]
The orbiting radio source thus lies at a radius of more than ten thousand times the Schwarzschild radius, so that nonrelativistic kinematics and Newtonian gravitation are justified.

4. A comet starts at infinity, goes around a relativistic star, and goes out to infinity. The impact parameter of the comet at infinity is $b$. The Schwarzschild radius of closest approach is $r_0$. What is the speed of the comet at closest approach as seen by an observer at that point?

**Solution:**

The comet trajectory in the equatorial plane $\theta = \pi/2$ is a timelike geodesic, with the effective action

$$S = \int ds \left[ c^2 \left( \frac{dt}{ds} \right)^2 \left( 1 - \frac{2R}{r} \right) - \frac{\left( \frac{dr}{ds} \right)^2}{1 - 2R/r} - r^2 \left( \frac{d\phi}{ds} \right)^2 \right]$$

The three constants of the motion are

$$r^2 \left( \frac{d\phi}{ds} \right) = B$$
$$\frac{dt}{ds} \left( 1 - \frac{2R}{r} \right) = A$$
$$c^2 \left( \frac{dt}{ds} \right)^2 \left( 1 - \frac{2R}{r} \right) - \frac{\left( \frac{dr}{ds} \right)^2}{1 - 2R/r} - r^2 \left( \frac{d\phi}{ds} \right)^2 = 1$$

We substitute the first two equations into the third one to obtain

$$c^2 A^2 = \left( \frac{dr}{ds} \right)^2 + \left( 1 - \frac{2R}{r} \right) \left[ 1 + \frac{B^2}{r^2} \right]$$

Then we use the relation

$$\frac{dr}{ds} = \frac{dr}{d\phi} \frac{d\phi}{ds}$$

and the first constant of the motion to obtain

$$\frac{c^2 A^2}{B^2} = \frac{1}{r^4} \left( \frac{dr}{d\phi} \right)^2 + \left( 1 - \frac{2R}{r} \right) \left( \frac{1}{B^2} + \frac{1}{r^2} \right)$$

We make the replacement $u = 1/r$ to get
\[ \frac{c^2A^2}{B^2} = \left( \frac{du}{d\phi} \right)^2 + u^2 (1 - 2R u) + \frac{1}{B^2} (1 - 2R u) \]

First we evaluate this expression at \( r = \infty \), where \( u = 0 \) and \( du/d\phi = 1/b \). We obtain

\[ \frac{c^2A^2}{B^2} = \frac{1}{b^2} + \frac{1}{B^2} \]

We also evaluate it at the distance of closest approach, \( u = u_0 = 1/r_0 \), for which \( du/d\phi = 0 \):

\[ \frac{c^2A^2}{B^2} = u_0^2 (1 - 2R u_0) + \frac{1}{B^2} (1 - 2R u_0) \]

We set these two expressions equal, and solve for \( 1/B^2 \):

\[ \frac{1}{B^2} = \frac{1}{2Ru_0} \left( u_0^2 (1 - 2R u_0) - \frac{1}{b^2} \right) \]

We then obtain

\[ \frac{c^2A^2}{B^2} = \frac{1 - 2Ru_0}{2Ru_0} \left( u_0^2 - \frac{1}{b^2} \right) \]

The velocity \( v_0 \) at the distance of closest approach is transverse, with magnitude

\[ v_T = r_0 \frac{d\phi}{dt} = r_0 \frac{d\phi}{ds} \frac{ds}{dt} = \frac{B/r_0}{A/(1 - 2R/r_0)} = \frac{B u_0 (1 - 2Ru_0)}{A} \]

Now we use the formula just obtained for \((cA/B)^2\) to get

\[ v_T = \frac{c \sqrt{2Ru_0 (1 - 2Ru_0)}}{\sqrt{1 - 1/(b^2u_0^2)}} \]

5. For the two dimensional metric

\[ ds^2 = (1 + gx)^2 (c dt)^2 - dx^2 \]
compute the Christoffel symbols and the Riemann tensor. In addition, find a full set of linearly independent Killing vectors.

**Solution:**
The components of the metric tensor are \( g_{00} = (1 + gx)^2 \) and \( g_{11} = -1 \). The Christoffel symbols are computed as follows:

\[
\begin{align*}
\Gamma_{bc}^a &= \frac{1}{2} g^{ad} \left[ \partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} \right] \\
\Gamma_{00}^1 &= \frac{1}{2} g^{11} \left[ -\partial_1 g_{00} \right] = g \ (1 + gx) \\
\Gamma_{10}^0 = \Gamma_{01}^0 &= \frac{1}{2} g^{00} \left[ -\partial_1 g_{00} \right] = \frac{g}{1 + gx}
\end{align*}
\]

The rest of the Christoffel symbols are zero. For the Riemann curvature tensor we have

\[
R_{abcd} = \partial_c \Gamma_{bd}^a - \partial_d \Gamma_{bc}^a + \Gamma_{ce}^a \Gamma_{bd}^e - \Gamma_{de}^a \Gamma_{bc}^e
\]

\[
R_{010} = \partial_0 \Gamma_{10}^0 = 0
\]

\[
R_{101} = \partial_1 \Gamma_{01}^0 - \Gamma_{10}^0 \Gamma_{10}^0 = \frac{g^2}{(1 + gx)^2} - \frac{g^2}{(1 + gx)^2} = 0
\]

All components of the Riemann tensor vanish, so that the space is flat. The covariant Killing vectors \((k_0(x_0, x), k_1(x_0, x))\) satisfy the following equations:

\[
\begin{align*}
D_0 k_b + D_b k_a &= 0 \\
\partial_a k_b + \partial_b k_a &= 2 \Gamma_{ab}^c k_c
\end{align*}
\]

We thus have the following three equations:

\[
\begin{align*}
\partial_1 k_1 &= \Gamma_{11}^c k_c = 0 \\
\partial_0 k_0 &= 2 \Gamma_{00}^1 k_1 = g (1 + gx) k_1 \\
\partial_0 k_1 + \partial_1 k_0 &= 2 \Gamma_{10}^0 k_0 = \frac{2g}{1 + gx} k_0
\end{align*}
\]

The first equation requires that \( k_1 \) is independent of \( x \); \( k_1 = f(x_0) \). The second equation then requires that
\[ \partial_0 k_0 = g(1 + gx)f(x_0) \]
\[ k_0 = h(x) + g(1 + gx)d(x_0) \]
\[ d'(x_0) = f(x_0) \]

We insert these results into the third equation to obtain

\[ d''(x_0) + h'(x) + g^2d(x_0) = \frac{2g}{1+gx}h(x) + 2g^2d(x_0) \]
\[ h'(x) - \frac{2g}{1+gx}h(x) = g^2d(x_0) - d''(x_0) \]

In the last relation, the left side depends only on \( x_0 \), whereas the right side depends only on \( x \). Therefore, they are each equal to a constant, \( \lambda \): For the left side:

\[ h' \frac{(1+gx)^2}{(1+gx)^3} - \frac{2gh}{(1+gx)^3} = \lambda \frac{(1+gx)^2}{(1+gx)^2} \]
\[ d \frac{h}{(1+gx)^2} \frac{d}{dx} \left( \frac{h}{(1+gx)^2} \right) = \lambda \frac{(1+gx)^2}{(1+gx)^2} \]
\[ h(1+gx)^2 = h_0 - \frac{\lambda}{g} \frac{1}{1+gx} \]
\[ h(x) = h_0(1+gx)^2 - \frac{\lambda}{g}(1+gx) \]

For the right side:

\[ g^2d(x_0) - d''(x_0) = \lambda \]
\[ d(x_0) = \frac{\lambda}{g^2} + Ae^{gx_0} + Be^{-gx_0} \]

We may then determine the Killing vectors:

\[ k_0 = h_0(1+gx)^2 + g(1+gx) (Ae^{gx_0} + Be^{-gx_0}) \]
\[ k_1 = gAe^{gx_0} - gBe^{-gx_0} \]
Note that the parameter $\lambda$ cancels out the expressions, and that there are three linearly independent Killing vectors. Here is a full set of linearly independent Killing vectors:

$$(k_0,k_1) = (1 + gx)^2 (1,0)$$

$$= (1 + gx) e^{gx_0} (1,g)$$

$$= (1 + gx) e^{-gx_0} (1,-g)$$

The transformation from this metric to two dimensional Minkowski space is given by the transformation $(x_0,x) \rightarrow (x'_0,x')$, where

$$x'_0 = \frac{1}{g+x} \sinh gx_0$$

$$x' = \frac{1}{g+x} \cosh gx_0$$

The infinitesimal changes in coordinates are given by

$$dx'_0 = dx \sinh gx_0 + dx_0 (1 + gx) \cosh gx_0$$

$$dx' = dx \cosh gx_0 + dx_0 (1 + gx) \sinh gx_0$$

Thus,

$$ds^2 = (dx'_0)^2 - (dx')^2 = (1 + gx)^2 (dx_0)^2 - dx^2$$

This result may be shown directly from the transformation formula for the Killing vectors:

$$k'_\nu = \frac{\partial u^\mu}{\partial u'^\nu} k'_\mu$$