Physics 403: Relativity

Takehome Final Examination

Due 07 May 2007

1. Suppose that a beam of protons (rest mass $m = 938\, MeV/c^2$ of total energy $E = 300\, GeV$ strikes a proton target at rest. Determine the largest mass of a particle X that could be produced in the following reaction:

$$p + p \rightarrow p + p + X$$

Solution:
The proton projectile (with mass m and total energy E) is incident upon a proton at rest. Thus, the total initial energy and momentum in the laboratory frame are given as follows:

$$\begin{aligned} E_i^{lab} &= E + mc^2 == 300.938\, GeV \\ c^2|\vec{p}_i^{\,lab}|^2 &= E^2 - m^2c^4 \end{aligned}$$

The final state consists of two protons (with energies E_1 and E_2, respectively) and the X-particle (with mass M_X and energy E_X). Thus, the final energy and momentum are

$$\begin{aligned} E_f^{lab} &= E_1 + E_2 + E_X \\ \vec{p}_f^{\,lab} &= \vec{p}_1 + \vec{p}_2 + \vec{p}_X \end{aligned}$$

Let us view this collision in the center of momentum frame, in which the total momentum is zero — before as well as after the collision.
We make use of Lorentz invariance of the quantity $E^2 - c^2p^2$ to determine the energy in the center of momentum frame:

$$(E_i^{cm})^2 = (E_i^{lab})^2 - c^2(\vec{p}_i^{\,lab})^2 = (E + mc^2)^2 - E^2 + m^2c^4 = 2mc^2(E + mc^2)$$

At the threshold for production of the X-particle, each of the three final state particles will be at rest in that frame. As a consequence

$$E_f^{cm} = 2mc^2 + m_X c^2$$

Setting these two energies equal, we obtain

$$
\begin{aligned}
E_i^{cm} &= E_f^{cm} \\
\sqrt{2mc^2(E+mc^2)} &= 2mc^2 + m_X c^2 \\
\sqrt{2\cdot 0.938\cdot 300.938}\, GeV &= 1.876\, GeV + m_X c^2 \\
23.780 &= 1.876 + m_X c^2 \\
m_X c^2 &= 21.884\, GeV
\end{aligned}
$$

2. A rocket ship accelerates directly away from the earth (radius R) with a constant acceleration g, as seen in the rest frame of the rocket.

- Calculate the angular size of the earth, as viewed from the rocket, expressed in terms of the proper time of flight on the rocket.
- Show that, as the proper time gets large, the angular size approaches the limiting value $2gR/c^2$.

Solution:
Let us restrict considerations to linear motion of the rocket along the x-direction, with the other spatial coordinates set to zero. The space time coordinates of the rocket ship in the frame of the earth at earth time t are $x^\mu = (ct, x)$. The four-velocity is given by

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}\right) = \gamma\left(c, \frac{dx}{dt}\right) = \gamma\,(c, v)$$

This four-velocity always satisfies the relation

$$u_\mu\, u^\mu = c^2$$

Let us define the four-acceleration as

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = \left(c\,\frac{d^2t}{d\tau^2}, v\frac{d^2x}{d\tau^2}\right)$$

It follows directly from the definition that

$$a_\mu\, u^\mu = 0$$

in every inertial frame. In the instananeous rest frame of the rocket (co-moving frame) the speed of the rocket is zero, so that $v^\mu = (c, 0)$ and $a^\mu = (0, g)$. we may calculate the frame-invariant quantity $a_\mu \, a^\mu = -g^2$.
Let us next calculate these two invariant quantities in the earth's frame:

$$\begin{aligned} a_\mu \, u^\mu = 0 &= = \gamma \left(c^2 \, \frac{d^2 t}{d\tau^2} - v \frac{d^2 x}{d\tau^2} \right) \\ a_\mu \, a^\mu = -g^2 &= \left(c \, \frac{d^2 t}{d\tau^2} \right)^2 - \left(\frac{d^2 x}{d\tau^2} \right)^2 \end{aligned}$$

Thus

$$\begin{aligned} c^2 \, \frac{d^2 t}{d\tau^2} &= v \, \frac{d^2 x}{d\tau^2} \\ \left(1 - \frac{v^2}{c^2}\right) \left(\frac{d^2 x}{d\tau^2} \right)^2 &= g^2 \\ \frac{d^2 x}{d\tau^2} = \frac{d}{d\tau} (\gamma \, v) &= \gamma \, g \\ \gamma \, \frac{dv}{d\tau} \left(1 + \gamma^2 \, \frac{v^2}{c^2} \right) &= \gamma \, g \\ \gamma^2 \, \frac{dv}{d\tau} &= g \\ \frac{dv}{1 - v^2/c^2} &= g \, d\tau \end{aligned}$$

We begin with $v = 0$ at proper time $\tau = 0$, so that

$$\begin{aligned} v &= c \tanh \frac{g\tau}{c} \\ \gamma &= \cosh \frac{g\tau}{c} \\ \frac{dx}{d\tau} &= \gamma v = c \sinh \frac{g\tau}{c} \end{aligned}$$

Since the rocket starts at $x_0 = R_e$ at proper time $\tau = 0$, its position at proper time τ is

$$x = \frac{c^2}{g}\left(\cosh\frac{g\tau}{c} - 1\right) + R_e$$

According to an observer inside the rocket, the distance to earth is Lorentz-contracted:

$$x' = \frac{x}{\gamma} = \frac{c^2(\cosh g\tau/c - 1) + g\,R_e}{g\cosh g\tau}$$

The angular size of the earth, as seen on the rocket, is θ', where

$$\tan\frac{\theta'}{2} = \frac{y'}{x'} = \frac{R}{x'} = \frac{g\,R_e\ \cosh g\tau/c}{c^2(\cosh g\tau/c - 1) + g\,R_e}$$

In the limit of large proper time τ we obtain the limiting angular size:

$$\tan\frac{\theta'}{2} \to \frac{g\,R_e}{c^2}$$

The rocketeer thus always has a (small) glimpse of the receding earth — at least in principle!

3. A black hole at the center of our galaxy has a radio source moving in a circular orbit about it with a visual radius θ of about 0.2 arc seconds, and with a period T of about 30 years. The galactic center is a distance D of about 10 kiloparsecs away from us. Using Newtonian gravity, determine the mass of the black hole.

<u>Solution</u>:

We shall use nonrelativistic kinematics and Newtonian gravity, under the assumption that the orbiting radio source (mass m and orbit radius r)is much lighter than the black hole (mass M), as well as far away from it. Consequently

$$\begin{aligned} F = \frac{GMm}{r^2} &= m\omega^2 r \\ \left[\frac{2\pi}{T}\right]^2 = \omega^2 &= \frac{GM}{r^2} \\ T^2 &= \frac{4\pi^2 r^3}{GM} \end{aligned}$$

This formula would also apply for the orbit of the earth about the sun (mass M_S), with its (average) radius r_E of $1\,AU \approx 1.5\times 10^{11}\ m$ and period $T_E = 1\,year$. That is

$$T_E^2 = \frac{4\pi^2 r_E^3}{GM_S}$$

Taking the ratio of these two expressions, we obtain

$$\begin{aligned} \left[\frac{T_E}{T}\right]^2 &= \left[\frac{r_E}{r}\right]^3 \frac{M}{M_S} \\ M &= M_S \left[\frac{T_E}{T}\right]^2 \left[\frac{r}{r_E}\right]^3 \end{aligned}$$

We may thus determine M (in solar masses) in terms of T in years and r in AU by using the fomula

$$M = T^2 r^3$$

The time T is 30 years. What is the distance r?
First we express the distance D to the galactic center in AU:

$$D = 10^4 \text{ parsec} \times \frac{1\,AU}{5 \times 10^{-6} \text{ parsec}} = 2 \times 10^9\,AU$$

Next, we obtain the visual radius θ in radians:

$$\theta = 0.2 \text{ arcsec} \times \frac{1 \text{ deg}}{3600 \text{ arcsec}} \times \frac{\pi\, rad}{1 \text{ deg}} = 9.7 \times 10^{-7} \text{ radians}$$

Thus the orbit radius is $r = D\theta = 1900\,AU$ and $M = 1900^3/30^2 = 8 \times 10^6$ solar masses. Since the solar mass is $2 \times 10^{30}\,kg$, we may express the mass as $M = 1.6 \times 10^{37}\,kg$.
The Schwarzschild radius of the black hole may then be computed

$$2R_0 = \frac{2GM}{c^2} = 2.4 \times 10^{10}\,m$$

Since $1\,AU = 1.5 \times 10^{11}\,m$, we obtain

$$r = 1900\,AU = 3 \times 10^{14}\,m$$

The orbiting radio source thus lies at a radius of more than ten thousand times the Schwarzschild radius, so that nonrelativistic kinematics and Newtonian gravitation are justified.

4. A comet starts at infinity, goes around a relativistic star, and goes out to infinity. The impact parameter of the comet at infinity is b. The Schwarzschild radius of closest approach is r_0. What is the speed of the comet at closest approach as seen by an observer at that point?
Solution:
The comet trajectory in the equatorial plane $\theta = \pi/2$ is a timelike geodesic, with the effective action

$$S = \int ds \left[c^2 \left(\frac{dt}{ds} \right)^2 \left(1 - \frac{2R}{r} \right) - \frac{\left(\frac{dr}{ds}\right)^2}{1 - 2R/r} - r^2 \left(\frac{d\phi}{ds} \right)^2 \right]$$

The three constants of the motion are

$$\begin{aligned} r^2 \left(\frac{d\phi}{ds} \right) &= B \\ \frac{dt}{ds} \left(1 - \frac{2R}{r} \right) &= A \\ c^2 \left(\frac{dt}{ds} \right)^2 \left(1 - \frac{2R}{r} \right) - \frac{\left(\frac{dr}{ds}\right)^2}{1 - 2R/r} - r^2 \left(\frac{d\phi}{ds} \right)^2 &= 1 \end{aligned}$$

We substitute the first two equations into the third one to obtain

$$c^2 A^2 = \left(\frac{dr}{ds} \right)^2 + \left(1 - \frac{2R}{r} \right) \left[1 + \frac{B^2}{r^2} \right]$$

Then we use the relation

$$\frac{dr}{ds} = \frac{dr}{d\phi} \frac{d\phi}{ds}$$

and the first constant of the motion to obtain

$$\frac{c^2 A^2}{B^2} = \frac{1}{r^4} \left(\frac{dr}{d\phi} \right)^2 + \left(1 - \frac{2R}{r} \right) \left(\frac{1}{B^2} + \frac{1}{r^2} \right)$$

We make the replacement $u = 1/r$ to get

$$\frac{c^2A^2}{B^2} = \left(\frac{du}{d\phi}\right)^2 + u^2\ (1-2R\,u) + \frac{1}{B^2}\ (1-2R\,u)$$

First we evaluate this expression at $r=\infty$, where $u=0$ and $du/d\phi = 1/b$. We obtain

$$\frac{c^2A^2}{B^2} = \frac{1}{b^2} + \frac{1}{B^2}$$

We also evaluate it at the distance of closest approach, $u = u_0 = 1/r_0$, for which $du/d\phi = 0$:

$$\frac{c^2A^2}{B^2} = u_0^2\ (1-2R\,u_0) + \frac{1}{B^2}\ (1-2R\,u_0)$$

We set these two expressions equal, and solve for $1/B^2$:

$$\frac{1}{B^2} = \frac{1}{2Ru_0}\ \left(u_0^2\ (1-2R\,u_0) - \frac{1}{b^2}\right)$$

We then obtain

$$\frac{c^2A^2}{B^2} = \frac{1-2Ru_0}{2Ru_0}\left(u_0^2 - \frac{1}{b^2}\right)$$

The velocity v_0 at the distance of closest approach is transverse, with magnitude

$$v_T = r_0\frac{d\phi}{dt} = r_0\ \frac{d\phi/ds}{dt/ds} = \frac{B/r_0}{A/(1-2R/r_0)} = \frac{B\,u_0\ (1-2Ru_0)}{A}$$

Now we use the formula just obtained for $(cA/B)^2$ to get

$$v_T = \frac{c\sqrt{2Ru_0\ (1-2Ru_0)}}{\sqrt{1-1/(b^2u_0^2)}}$$

5. For the two dimensional metric

$$ds^2 = (1+gx)^2(cdt)^2 - dx^2$$

compute the Christoffel symbols and the Riemann tensor. In addition, find a full set of linearly independent Killing vectors.
Solution:
The components of the metric tensor are $g_{00} = (1+gx)^2$ and $g_{11} = -1$.
The Christoffel symbols are computed as follows:

$$\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2} g^{ad} \left[\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\right] \\
\Gamma^1_{00} &= \frac{1}{2} g^{11} \left[-\partial_1 g_{00}\right] = g\,(1+gx) \\
\Gamma^0_{10} = \Gamma^0_{01} &= \frac{1}{2} g^{00} \left[-\partial_1 g_{00}\right] == \frac{g}{1+gx}
\end{aligned}$$

The rest of the Christoffel symbols are zero. For the Riemann curvature tensor we have

$$\begin{aligned}
R^a_{bcd} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc} \\
R^1_{010} &= -\partial_0 \Gamma^1_{01} = 0 \\
R^0_{101} &= -\partial_1 \Gamma^0_{10} - \Gamma^0_{10}\Gamma^0_{10} = \frac{g^2}{(1+gx)^2} - \frac{g^2}{(1+gx)^2} = 0
\end{aligned}$$

All components of the Riemann tensor vanish, so that the space is flat. The covariant Killing vectors $(k_0(x_0,x), k_1(x_0,x))$ satisfy the following equations:

$$\begin{aligned}
D_a k_b + D_b k_a &= 0 \\
\partial_a k_b + \partial_b k_a &= 2\Gamma^c_{ab} k_c
\end{aligned}$$

We thus have the following three equations:

$$\begin{aligned}
\partial_1 k_1 &= \Gamma^c_{11} k_c = 0 \\
\partial_0 k_0 &= 2\Gamma^1_{00} k_1 = g(1+gx) k_1 \\
\partial_0 k_1 + \partial_1 k_0 &= 2\Gamma^0_{10} k_0 = \frac{2g}{1+gx} k_0
\end{aligned}$$

The first equation requires that k_1 is independent of x; $k_1 = f(x_0)$. The second equation then requires that

$$
\begin{aligned}
\partial_0 k_0 &= g(1+gx)f(x_0) \\
k_0 &= h(x)+g(1+gx)d(x_0) \\
d'(x_0) &= f(x_0)
\end{aligned}
$$

We insert these results into the third equation to obtain

$$
\begin{aligned}
d''(x_0)+h'(x)+g^2d(x_0) &= \frac{2g}{1+gx}h(x)+2g^2d(x_0) \\
h'(x)-\frac{2g}{1+gx}h(x) &= g^2d(x_0)-d''(x_0)
\end{aligned}
$$

In the last relation, the left side depends only on x_0, whereas the right side depends only on x. Therefore, they are each equal to a constant, λ: For the left side:

$$
\begin{aligned}
\frac{h'}{(1+gx)^2}-\frac{2gh}{(1+gx)^3} &= \frac{\lambda}{(1+gx)^2} \\
\frac{d}{dx}\left[\frac{h}{(1+gx)^2}\right] &= \frac{\lambda}{(1+gx)^2} \\
\frac{h}{(1+gx)^2} &= h_0-\frac{\lambda}{g}\frac{1}{1+gx} \\
h(x) &= h_0(1+gx)^2-\frac{\lambda}{g}(1+gx)
\end{aligned}
$$

For the right side:

$$
\begin{aligned}
g^2d(x_0)-d''(x_0) &= \lambda \\
d(x_0) &= \frac{\lambda}{g^2}+Ae^{gx_0}+Be^{-gx_0}
\end{aligned}
$$

We may then determine the Killing vectors:

$$
\begin{aligned}
k_0 &= h_0\,(1+gx)^2+g\,(1+gx)\,\left(Ae^{gx_0}+Be^{-gx_0}\right) \\
k_1 &= gAe^{gx_0}-gBe^{-gx_0}
\end{aligned}
$$

Note that the parameter λ cancels out the expressions, and that there are three linearly independent Killing vectors. Here is a full set of linearly independent Killing vectors:

$$\begin{aligned}
(k_0, k_1) &= 1 + gx)^2 \; (1,0) \\
&= (1+gx) \, e^{gx_0} \; (1,g) \\
&= (1+gx) \, e^{-gx_0} \; (1,-g)
\end{aligned}$$

The transformtion from this metric to two dimensional Minkowski space is given by the transformation $(x_0, x) \to (x'_0, x')$, where

$$\begin{aligned}
x'_0 &= (1/g + x) \; \sinh gx_0 \\
x' &= (1/g + x) \; \cosh gx_0
\end{aligned}$$

The infinitesimal changes in coordinates are given by

$$\begin{aligned}
dx'_0 &= dx \; \sinh gx_0 + dx_0 \; (1+gx) \; \cosh gx_0 \\
dx' &= dx \; \cosh gx_0 + dx_0 \; (1+gx) \; \sinh gx_0
\end{aligned}$$

Thus,

$$ds^2 = (dx'_0)^2 - (dx')^2 = (1+gx)^2 \; (dx_0)^2 - dx^2$$

This result may be shown directly from the transformation formula for the Killing vectors:

$$k_\nu = \frac{\partial u'^\mu}{\partial u^\nu} \; k'_\mu$$