Physics 403: Relativity

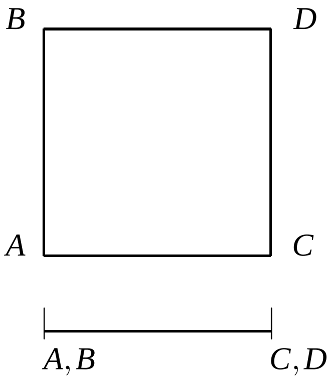
Homework Assignment 1

Due 29 January 2007

1. The visual image of an object passing by an observer can be constructed (in principle) by recording the time of arrival of light rays and their direction, to determine the point from which they were send. Consider the four vertices of a square (A, B, C, D) moving with velocity v lying in the plane of the square and in the direction AB, and passing by an observer also in the plane of the square. Show that the image of the passing square appears rotated (rather than Lorentz contracted). Explain the apparent contradiction.

<u>Solution</u>:

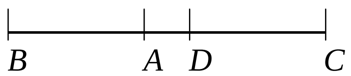
Here is a picture of the square of length L at rest.



Square at Rest

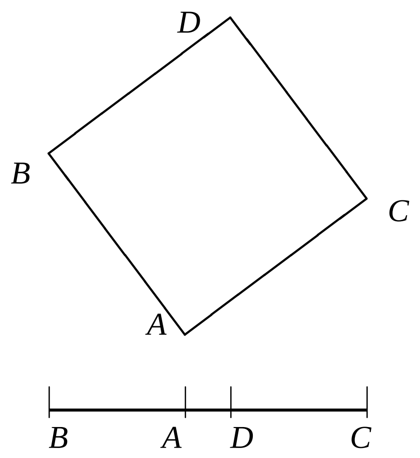
When viewed from a great distance, the points A, B and C, D appear at the same point, since one's "depth perception" is missing. This is indicated on the horizontal line below the square

When the square moves to the right at velocity v, the distances AC and BC become contracted to $\sqrt{1-\beta^2}L$. In addition, points A and C are closer to the observer than points B and D. Consequently, we see those points at a distance βL to the right on the horizontal line, as shown below:



Square in Motion

We might imagine that the latter image would be produced by a tilted square, as shown:



Rotated Square

That's how we would interpret this image, based on the light signals that we receive at a particular time.

2. The Lorentz transformation equations have no meaning if the relative velocity of two frames is greater than the speed of light. This is taken to imply that mass, energy, and information (messages) cannot be moved from place to place than the speed of light. In the light of this, consider the "Scissors Paradox". A very long straight rod, which is inclined at an angle ϕ to the x-axis, moves downward with speed v, which is close to the speed of light. Find the speed v' of the point of intersection of the lower edge of the rod with the x-axis. Can this speed be greater than the speed of light? Can it be used to transmit a message to somebody far out on the x-axis?
Solution: Suppose that the rod is moving downward with speed v, being tilted at a small angle ϕ relative to the horizontal. As the rod crosses the x-axis, the speed with which it its intersection with the x-axis moves, V, satisfies the relation

$$\frac{v}{V} = \tan\phi$$

Consequently, for small ϕ we obtain $V \approx v/\phi$. This can be substantially greater than the speed of light. There is no contradiction with relativity, since no physical signal is associated with this crossing.

3. Consider a transformation of the longitudinal position x and time t from one

inertial frame to another moving with speed v, subject to the following requirements.

- Linearity:

$$\begin{aligned} x' &= a(v)(x - vt) \\ t' &= b(v)x + e(v)t \end{aligned}$$

- $$\begin{aligned} x &= a(v)(x' + vt') \\ t &= b(v)x' + f(v)t' \end{aligned}$$

- The composition of two such transformations is also a transformation of the same form.

Show that the Galilean transform and the Lorentz transform are the only possibilities consistent with these requirements.
Solution:
Let us express the original transformation as

$$\begin{aligned} x' &= a(v)(x - vt) \\ t' &= b(v)(t + ve(v)x) \end{aligned}$$

Let us substitute these expressions into the first of the inverse equations:

$$\begin{aligned} x &= a(v)(x' + vt') = a(v)\left[a(v)(x - vt) + +vbv(x + ve(v)t)\right] \\ &= a(v)\left[(av - v^2 b(v)e(v))x + vt(b(v) - a(v))\right] \end{aligned}$$

For consistency we require that $a(v) = b(v)$ and $1 = a^2(v)(1 + v^2 e(v))$. Thus, we may write the original transformation as

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 + v^2 e(v)}}(x - vt) \\ t' &= \frac{1}{\sqrt{1 + v^2 e(v)}}(t + ve(v)x) \end{aligned}$$

Consider the composition of two Lorentz boosts:

$$\begin{aligned}
x' &= \frac{1}{\sqrt{1+v_1^2 e(v_1)}}(x - v_1 t) \\
t' &= \frac{1}{\sqrt{1+v_1^2 e(v_1)}}(t + v_1 e(v_1) x) \\
x'' &= \frac{1}{\sqrt{1+v_2^2 e(v_2)}}(x' - v_2 t') \\
t'' &= \frac{1}{\sqrt{1+v_2^2 e(v_2)}}(t' + v_2 e(v_2) x')
\end{aligned}$$

Substituting the first two relations into the third one, we obtain

$$\begin{aligned}
x'' &= \frac{1}{\sqrt{1+v_2^2 e(v_2)}}\frac{1}{\sqrt{1+v_1^2 e(v_1)}}\left[(x - v_1 t) - v_2(t + v_1 e(v_1) x)\right] \\
&= \frac{1}{\sqrt{1+v_2^2 e(v_2)}}\frac{1}{\sqrt{1+v_1^2 e(v_2)}}\left[(1 - v_2 v_1 e(v_1))x - (v_1 + v_2)t\right]
\end{aligned}$$

and

$$\begin{aligned}
t'' &= \frac{1}{\sqrt{1+v_2^2 e(v_2)}}\frac{1}{\sqrt{1+v_1^2 e(v_1)}}\left[(t + v_1 e(v_1) x) + v_2 e(v_2)(x - v_1 t)\right] \\
&= \frac{1}{\sqrt{1+v_2^2 e(v_2)}}\frac{1}{\sqrt{1+v_1^2 e(v_1)}}\left[(1 - v_2 v_1 e(v_2))t + (v_1 e(v_1) + v_2 e(v_2))x\right]
\end{aligned}$$

These equations must therefore have the form

$$\begin{aligned}
x'' &= \frac{1}{\sqrt{1+v^2 e(v)}}(x - vt) \\
t'' &= \frac{1}{\sqrt{1+v^2 e(v)}}(t + v e(v) x)
\end{aligned}$$

where v is the composite velocity. Thus, for consistency we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{1+v^2e(v)}} &= \frac{1}{\sqrt{1+v_2^2e(v_2)}}\frac{1}{\sqrt{1+v_1^2e(v_1)}}(1-v_2v_1e(v_1)) \\
&= \frac{1}{\sqrt{1+v_2^2e(v_2)}}\frac{1}{\sqrt{1+v_1^2e(v_1)}}(1-v_2v_1e(v_2)) \\
v &= \frac{v_1+v_2}{1-v_2v_1e(v_1)} \\
e(v)v &= \frac{e(v_1)v_1-e(v_2)v_2}{1-v_2v_1e(v_2)}
\end{aligned}
$$

We conclude that $e(v_1)=e(v_2)$, so that e must be a constant independent of velocity. The velocity of composition is

$$
v=\frac{v_1+v_2}{1-ev_1v_2}
$$

Using this velocity, we calculate the corresponding γ factor:

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1+v^2e}} = \frac{1-ev_1v_2}{\sqrt{(1-ev_1v_2)^2+e(v_1+v_2)^2}} \\
&= \frac{1-ev_1v_2}{\sqrt{1+e^2v_1^2v_2^2+ev_1^2+ev_2^2}} \\
&= \frac{1-ev_1v_2}{\sqrt{1+ev_1^2}\sqrt{1+ev_2^2}}
\end{aligned}
$$

Thus, the transformation must be

$$
\begin{aligned}
x'' &= \frac{1}{\sqrt{1+ev^2}}(x-vt) \\
t'' &= \frac{1}{\sqrt{1+ev^2}}(t+evx)
\end{aligned}
$$

For the parameter $e=0$ we obtain the Galilei transformation, whereas for e negative; $e=-1/c^2$; we obtain the Lorentz transformation for velocity c. The case

of positive e corresponds to rotation of the variables $(x, t/\sqrt{e})$ through an angle θ, with $\tan\theta = \sqrt{e}v$.
This result was originally obtained by P. Frank and H. Rohte in 1911. It represents the first time that Lorentz transformations were treated as a symmetry group.

4. A certain man walks very fast – so fast that the relativistic length contraction makes him very thin. In the street he has to pass over a grid. A man standing at the grid fully expects the fast thin man to fall through the holes in the grid. Yet to the fast man he himself has his usual size, whereas the grid has the relativistic contraction. To him the holes in the grid are much narrower than to the stationary man, and he certainly does not expect to fall through them. Which man is correct? Hint: The answer hinges on the lack of rigidity in relativity.
Solution:
The problem may be idealized as a very thin rod sliding lengthwise on a flat table, with a hole in its path. It the Lorentz contraction factor is large, then in the frame of the table the rod is much shorter than the hole, and will drop into it. Assume that in that frame the rod remains essentially horizontal, without "tipping", as it descends into the hole. For small vertical velocities the rod will fall with the usual acceleration g.
In the frame of the rod, the hole is Lorentz-contracted and the rod cannot possibly fit into it. However, one may perform a Lorentz transform of the world line of the front of the rod, and show that it wil "droop" over the edge of the table in that frame. That is, the rod cannot be rigid.
This problem was discussed by W. Rindler, American Journal of Physics 29, 363 (1961).

5. Consider the following sequence of infinitesimal Lorentz boosts:

- A Lorentz boost of speed v_1 in the x-direction.
- A Lorentz boost of speed v_2 in the y-direction.
- A Lorentz boost of speed v_1 in the $-x$-direction.
- A Lorentz boost of speed v_2 in the $-y$-direction.

Show that the net effect of this sequence is a spatial rotation by an infinitesimal angle θ in the z-direction. Determine θ. Hint: Keep infinitesimals only up to second order.
Let the coordinates be represented by a column vector, so that the Lorentz transforms correspond to a four-dimensional matrix $L(\vec{v})$. The first two Lorentz boosts

are represented in terms of the parameters $\beta_{1,2} = v_{1,2}/c$ and $\gamma_{1,2} = 1/\sqrt{1-\beta_{1,2}^2}$ by these matrices

$$L(v_1\hat{i}) = \begin{bmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ \beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L(v_2\hat{j}) = \begin{bmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ \beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We keep terms to second order in $\beta_{1,2}$, so that $\gamma_{1,2} \approx 1+\beta_{1,2}^2/2$, so that the product of these transformations is

$$\begin{aligned} L(v_2\hat{j})L(v_1\hat{i}) &= \begin{bmatrix} 1+\beta_2^2/2 & 0 & -\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \beta_2 & 0 & 1+\beta_2^2/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\beta_1^2/2 & -\beta_1 & 0 & 0 \\ \beta_1 & 1+\beta_1^2/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+\beta_1^2/2 & 0 & 0 & -\beta_1 \\ \beta_1\beta_2 & 1+\beta_2^2/2 & 0 & -\beta_2 \\ 0 & 0 & 1 & 0 \\ -\beta_1 & -\beta_2 & 0 & 1+(\beta_1^2+\beta_2^2)/2 \end{bmatrix} \end{aligned}$$

The composition matrix for the next two Lorentz transformations, $L(-v_2\hat{j})L(-v_1\hat{i})$ is obtained from this one by making the replacements $\beta_{1,2} \to -\beta_{1,2}$, so that for the full transformation we obtain

$$\begin{aligned} & L(-v_2\hat{j})L(-v_1\hat{i})L(v_2\hat{j})L(v_1\hat{i}) = \\ & \begin{bmatrix} 1+\beta_1^2/2 & 0 & 0 & -\beta_1 \\ \beta_1\beta_2 & 1+\beta_2^2/2 & 0 & -\beta_2 \\ 0 & 0 & 1 & 0 \\ -\beta_1 & -\beta_2 & 0 & 1+(\beta_1^2+\beta_2^2)/2 \end{bmatrix} \\ \times & \begin{bmatrix} 1+\beta_1^2/2 & 0 & 0 & \beta_1 \\ \beta_1\beta_2 & 1+\beta_2^2/2 & 0 & \beta_2 \\ 0 & 0 & 1 & 0 \\ \beta_1 & \beta_2 & 0 & 1+(\beta_1^2+\beta_2^2)/2 \end{bmatrix} \\ = & \begin{bmatrix} 1 & -\beta_1\beta_2 & 0 & 0 \\ \beta_1\beta_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

This corresponds to a rotation about the z-axis through the infinitesimal angle $\theta = \beta_1\beta_2$.

6. Astronauts in a spaceship travelling on a straight path past the earth at speed $v = c/2$ wish to tune in WGN radio at 720 kHz. To what frequency should they tune at the instant when the ship is closest to earth?
Solution:
This is a question involving the transverse Doppler Shift, with $\beta = v/c = 0.5$.

$$\nu' = \frac{\nu}{\sqrt{1-\beta^2}} = \frac{720\,KHz}{\sqrt{3/4}} = 831\,kHz$$