Physics 403: Relativity

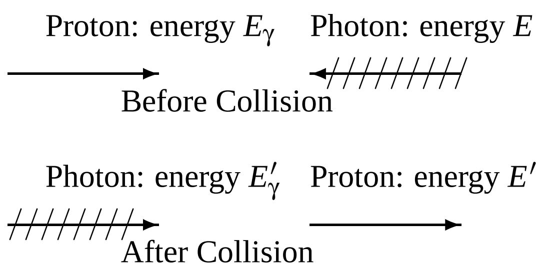
Homework Assignment 2

Due 12 February 2007

1. Inverse Compton scattering occurs whenever a photon scatters off a particle moving with a speed very nearly equal to that of light. Suppose that a particle of rest mass m and total energy $\mathcal{E}$ collides head on with a photon of energy $\mathcal{E}_\gamma$. Show that the scattered photon has energy

$$\mathcal{E}\left[1+\frac{m^2c^4}{4\mathcal{E}\mathcal{E}_\gamma}\right]^{-1}$$

Ultra-high energy cosmic rays have energies up to 10^{20} eV. How much energy can a cosmic ray proton transfer to a microwave background photon?
Solution:
The collinear collision is pictured below:



The equations of conservation of energy and momentum are

$$\begin{aligned} E+E_\gamma &= E'+E'_\gamma \\ p-p_\gamma &= p'+p'_\gamma \end{aligned}$$

Since the initial and final photon are massless, we have $E_\gamma = cp_\gamma$ and $E'_\gamma = cp'_\gamma$. Let us solve for the final energy and momentum of the proton:

$$\begin{aligned} E' &= E + E_\gamma - E'_\gamma \\ cp' &= cp - E_\gamma - E'_\gamma \end{aligned}$$

Thus

$$\begin{aligned} E'^2 - (cp')^2 &= (E + E_\gamma - E'_\gamma)^2 - (cp - E_\gamma - E'_\gamma)^2 \\ &= E^2 - (cp)^2 + 2E(E_\gamma - E'_\gamma) + 2cp(E_\gamma + E'_\gamma) - 4E_\gamma E'_\gamma \end{aligned}$$

Let us make use of the energy-momentum relations $E'^2 - (cp')^2 = m^2c^4 = E^2 - (cp)^2$ to obtain

$$E'_\gamma \left[E - cp + 2E_\gamma\right] = E_\gamma \left[E + cp\right]$$

In the extreme relativistic limit, we make the replacement $cp \to E$ in this expression to obtain

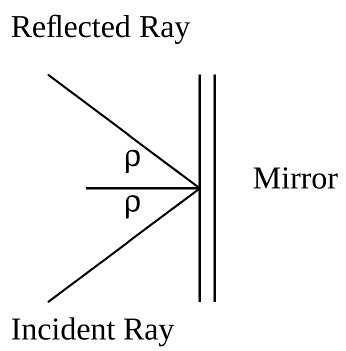
$$E'_\gamma = \frac{EE_\gamma}{E_\gamma + m^2c^4/(4E)}$$

As a crude estimate, that $mc^2 = 10^9\ eV$ and $E_\gamma = 10^{-4}\ eV$, corresponding to microwave black body radiation at a temperature of a few Kelvins. For an incident cosmic ray proton of energy $10^{20}\ eV$, we obtain $E'_\gamma \approx 4 \times 10^{18}\ eV$. Consequently, the cosmic ray loses 4% of its energy in such a collision. It is felt that this effect is responsible for the decrease in the number of high energy cosmic rays at about this energy.

2. A ray of light is reflected from a plane mirror, which moves in the direction of its normal with velocity v. Prove that the angles of incidence and reflection, θ and ϕ, respectively, are related by the formula

$$\frac{\sin\theta}{\sin\phi} = \frac{\cos\theta \pm v/c}{\cos\phi \mp v/c}$$

<u>Solution</u>:
The incident and reflected rays are shown in the rest frame of the mirror.

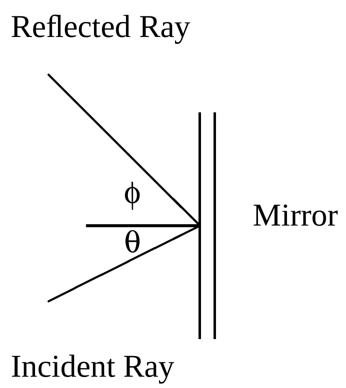


Rest Frame of Mirror

Their four-vector momenta, $(k_i)^\mu$ and $(k_f)^\mu$, are given in that frame as follows:

$$
\begin{aligned}
(k_i)^\mu &= (k, k\cos\rho, k\sin\rho, 0) \\
(k_f)^\mu &= (k, -k\cos\rho, k\sin\rho, 0)
\end{aligned}
$$

In the moving frame, these rays are shown below:



Moving Observer

Their four-vector momenta, $(k_i')^\mu$ and $(k_f')^\mu$, are given in that frame as follows:

$$
\begin{aligned}
(k_i')^\mu &= (k_i', k_i'\cos\theta, k_i'\sin\theta, 0) \\
(k_f')^\mu &= (k_f', -k_f'\cos\rho, k_f'\sin\phi, 0)
\end{aligned}
$$

The respective four-momenta in the two frames are related by the Lorentz transrform. It is convenient to apply the inverse transforms. We do so for the x and y components of the incident four-momentum:

$$
\begin{aligned}
(k_i)^1 &= \gamma\left((k_i')^1 + \beta (k_i')^0\right) \\
k\cos\rho &= \gamma k_i' \left(\cos\theta + \beta\right)
\end{aligned}
$$

$$
\begin{aligned}
(k_i)^2 &= (k_i')^2 \\
k\sin\rho &= k_i' \sin\theta
\end{aligned}
$$

Taking the ratio of these equations, we obtain

$$\tan\rho = \frac{\sin\theta}{\gamma\left(\cos\theta + \beta\right)}$$

Now we consider the x and y components of the reflected beam:

$$
\begin{aligned}
(k_f)^1 &= \gamma\left((k_f')^1 + \beta (k_f')^0\right) \\
-k\cos\rho &= \gamma k_f' \left(-\cos\phi + \beta\right)
\end{aligned}
$$

$$
\begin{aligned}
(k_f)^2 &= (k_f')^2 \\
k\sin\rho &= k_f' \sin\phi
\end{aligned}
$$

Taking the ratio of these equations, we obtain

$$\tan\rho = \frac{\sin\phi}{\gamma(\cos\phi - \beta)}$$

Since the two expressions for $\tan\rho$ must be identical, we obtain

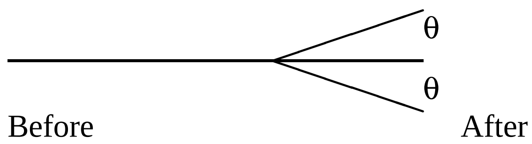
$$\frac{\sin\theta}{\gamma\left(\cos\theta + \beta\right)} = \frac{\sin\phi}{\gamma\left(\cos\phi - \beta\right)}$$

The factors of γ cancel out, so that

$$\frac{\sin\theta}{\cos\theta+\beta} = \frac{\sin\phi}{\cos\phi-\beta}$$

The result is thus established.

3. A particle of mass m and total energy E decays into two massless particles that travel symmetrically with respect to the direction of motion of the original particle. Determine the angle θ of the decay products with respect to the original direction. As particular cases, consider the non-relativistic case and the extreme relativistic case.
<u>Solution</u>:
Here is a picture of the situation before and after the decay process:



Decay of Massive Particle into Two Photons

Let the incident particle have initial energy $\mathcal{E}$ and momentum $\vec{p}$ to the right. The final photons have momenta $\vec{p}_\pm$ and energies $\mathcal{E}\pm = cp_\pm$:

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_+ + \mathcal{E}_- \\ \vec{p} &= \vec{p}_+ + \vec{p}_- \end{aligned}$$

It follows from symmetry that $\mathcal{E}_+ = \mathcal{E}_-$, and that the momenta $\vec{p}_\pm$ are of equal magnitude. Thus

$$\begin{aligned} \mathcal{E} &= 2\mathcal{E}_+ \\ p &= 2p_+\cos\theta \end{aligned}$$

As a consequence, with v the speed of the incident particle, we obtain

$$\frac{v}{c} = \frac{cp}{\mathcal{E}} = \frac{cp_+\cos\theta}{\mathcal{E}_+} = \cos\theta$$

Note that, when the speed of the incident particle is small compared with the velocity of light, the opening angle between the two photons (2θ) is close to 90°. As the speed of the incident particle increases, that opening angle becomes smaller, approaching zero in the extreme relativistic limit.

4. Calculate the area of a circle of radius r (distance from center to circumference) in the two-dimensional geometry that is a sphere of radius a. Show that this reduces to πr^2 when $r \ll a$.
<u>Solution</u>:
Choose a spherical polar coordinate system with the polar axis (North pole) at the center of the circle. The radius of the sphere is $r = a\,\theta_0$, where θ_0 is the polar angle subtended by the perimeter of the circle. The surface area inside the (curved) circle is

$$A = a^2 \int_0^{\theta_0} d\theta\ \sin\theta \int_0^{2\pi} d\phi = 2\pi a^2 (1 - cos\theta_0) = 2\pi a^2 (1 - cos\frac{r}{a})$$

From the relation

$$\cos\theta_0 = 1 - \frac{\theta_0^2}{2} + \frac{\theta_0^4}{24} - \cdots$$

we obtain

$$A = 2\pi a^2 \left(\frac{\theta_0^2}{2} - \frac{\theta_0^4}{24} \right) = \pi r^2 \left(1 - \frac{\theta_0^2}{12} \right) = \pi r^2 \left(1 - \frac{r^2}{12a^2} \right)$$

When $r = 100\,m$, the area is approximately $\pi \times 10^4\,m^2 = \pi$ hectares, differing from that value by about 2×10^{-11}. When $= 100\,km$, the deviation is approximately 2×10^{-5}.

5. Consider a particle with four-momentum p and an observer with four-velocity $u = (\gamma c, \gamma\vec{v})$. Show that, if the particle goes through the observer's laboratory, the magnitude of the three-momentum measured is

$$|\vec{p}'|^2 = \left(\frac{p \cdot u}{c} \right)^2 - p \cdot p$$

<u>Solution</u>:
Suppose that the observer is moving in the x-direction with speed v, so that $u^\mu = \gamma(c, v, 0, 0) = \gamma c(1, \beta, 0, 0)$.
The four-momentum of the particle in the initial frame is $p^\mu = (p^0, \vec{p})$, so that

$$
\begin{aligned}
p \cdot p = p^\mu p_\mu &= (p^0)^2 - \vec{p} \cdot \vec{p} = p_0^2 - p_x^2 - p_y^2 - p_z^2 \\
p \cdot u = p^\mu u_\mu &= p^0 u^0 - \vec{p} \cdot \vec{u} = p^0 u^0 - p_x u_x = \gamma c (p^0 - \beta p_x)
\end{aligned}
$$

We thus obtain

$$
\left(\frac{p \cdot u}{c} \right)^2 = \gamma^2 (p^0 - \beta p_x)^2
$$

Consequently

$$
\begin{aligned}
\left(\frac{p \cdot u}{c} \right)^2 - p \cdot p &= \gamma^2 (p^0 - \beta p_x)^2 - ((p^0)^2 - p_x^2 - p_y^2 - p_z^2) \\
&= (\gamma^2 - 1)(p^0)^2 - 2\gamma^2 \beta p^0 p_x + (1 + \beta^2 \gamma^2) p_x^2 + p_y^2 + p_z^2 \\
&= \gamma^2 (p_x^2 - 2\beta p_x p^0 + \beta^2 (p^0)^2) + p_y^2 + p_z^2 \\
&= \gamma^2 (p_x - \beta p^0)^2 + p_y^2 + p_z^2
\end{aligned}
$$

The components of spatial momentum in the frame of the moving observer are

$$
\begin{aligned}
p'_x &= \gamma (p_x - \beta p^0) \\
p'_y &= p_y \\
p'_z &= p_z
\end{aligned}
$$

so that

$$
|\vec{p}'|^2 = \gamma^2 (p_x - \beta p^0)^2 + p_y^2 + p_z^2
$$

The result is thus established.