Physics 403: Relativity

Homework Assignment 3

Due 26 February 2007

1. In an inertial frame, show that the 3-acceleration of a charged particle in an electromagnetic field is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{q}{\gamma m}\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B} - \frac{1}{c^2}(\vec{v}\cdot\vec{E})\vec{v}\right]$$

Solution:
Let us consider the relativistic generalization of the Lorentz force relation:

$$\begin{aligned}
\frac{d\vec{p}}{dt} &= q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}) \\
\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} &= \frac{d\vec{v}}{dt}\,\frac{m}{\sqrt{1-v^2/c^2}} + \vec{v}\,\frac{d}{dt}\frac{m}{\sqrt{1-v^2/c^2}} \\
&= \frac{d\vec{v}}{dt}\,\frac{m}{\sqrt{1-v^2/c^2}} + \frac{\vec{v}}{c^2}\,\frac{d\mathcal{E}}{dt}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \frac{d}{dt}\frac{m}{\sqrt{1-v^2/c^2}} \\
&= \frac{m}{(1-v^2/c^2)^{3/2}}\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right)
\end{aligned}$$

It follows from the relation $\vec{p} = \mathcal{E}\vec{v}/c^2$ that

$$\begin{aligned}
\vec{v}\cdot\frac{d\vec{p}}{dt} &= q(\vec{v}\cdot\vec{E}) \\
&= \frac{d\mathcal{E}}{dt}\frac{v^2}{c^2} + \frac{\mathcal{E}}{c^2}\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right) \\
&= \frac{d\mathcal{E}}{dt}\frac{v^2}{c^2} + (1-v^2/c^2)\frac{d\mathcal{E}}{dt} \\
&= \frac{d\mathcal{E}}{dt}
\end{aligned}$$

We insert this into the first relation to obtain

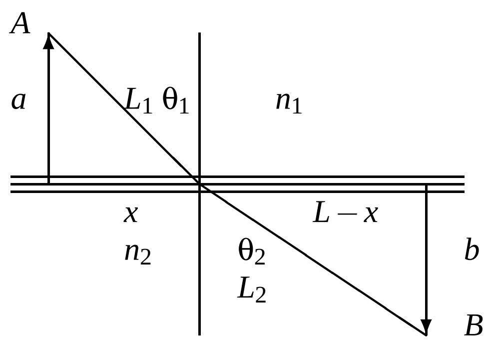
$$
\begin{aligned}
\frac{d\vec{v}}{dt}\frac{m}{\sqrt{1-v^2/c^2}}+\frac{q\vec{v}}{c^2}\left(\vec{v}\cdot\vec{E}\right) &= q\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right) \\
\frac{d\vec{v}}{dt} &= \frac{q}{\gamma m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}-\vec{v}\,\frac{\vec{v}\cdot\vec{E}}{c^2}\right)
\end{aligned}
$$

The result is thus established.

2. Fermat's principle states that a light ray travels from point A to point B along the path corresponding to mininum elapsed time. Use Fermat's principle to show that, when a light ray travels from a point A in a medium of index of refraction n_1 to a point B in a medium of index of refraction n_2, the angles between path and the normals to the interface obey Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

<u>Solution</u>:
In each of the media the shortest path corresponds to straight lines. Thus, the trajectory of the light ray from point A to point B consists of two straight lines, as shown:



In the top medium, the light ray travels a distance $L_1=\sqrt{a^2+x^2}$, whereas in the bottom medium it travels a distance $L_2=\sqrt{b^2+(L-x)^2}$.
In the top medium, with index of refraction n_1, the velocity of light is c/n_1, and the light ray travels a distance L_1 in time $t_1 = n_1 L_1/c$.

For the bottom medium, the index of refraction is n_1, the velocity of light is c/n_2, and the light ray travels a distance L_2 in time $t_2 = n_2 L_2/c$. Thus, the total time of travel from A to B is

$$T = \frac{n_1}{c}\sqrt{a^2+x^2} + \frac{n_2}{c}\sqrt{b^2+(L-x)^2}$$

We choose x by requiring that the time T be a minimum, so that

$$\frac{dT}{dx} = \frac{n_1}{c}\frac{x}{\sqrt{a^2+x^2}} - \frac{n_2}{c}\frac{L-x}{\sqrt{b^2+(L-x)^2}} = 0$$

Thus

$$\begin{aligned} n_1\frac{x}{\sqrt{a^2+x^2}} &= n_2\frac{L-x}{\sqrt{b^2+(L-x)^2}} \\ n_1\sin\theta_1 &= n_2\sin\theta_2 \end{aligned}$$

The result is established.

3. Obtain the general expression for the electric field of a uniformly moving charge by making a Lorentz transformation of the static Coulomb field.
Solution: In the rest frame of the charged particle, the electric field at point P is

$$\vec{E} = \frac{q}{r^3}\vec{r}$$

where $\vec{r} = (x,y,z)$ is the vector from the charge to the point P.
Let us suppose that we do a Lorentz boost in the z-direction. The longitudinal electric field is unchanged, whereas the transverse fields are multiplied by a factor γ:

$$\begin{aligned} E'_x &= \gamma E_x = \frac{\gamma q x}{r^3} \\ E'_y &= \gamma E_y = \frac{\gamma q y}{r^3} \\ E'_z &= E_z = \frac{qz}{r^3} \end{aligned}$$

However, this field in the new inertial frame is still expressed in terms of coordinates in the old inertial frame.

The world line of the point P in the old inertial frame is $x^\mu = (ct, x, y, z)$. This point is transformed into $(x')^\mu = (ct', x', y', z')$:

$$\begin{aligned} t' &= \gamma(t - vz/c^2) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \end{aligned}$$

We wish to find the spatial coordinates at $t' = 0$, or $t = vz/c^2$. We obtain

$$\begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma z' \end{aligned}$$

We must also express the distance r in the transformed coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2\gamma^2 \\ &= \gamma^2\left((1 - v^2/c^2)\,(x'^2 + y'^2) + z'^2\right) \\ &= \gamma^2 r'^2 (1 - v^2 \sin^2\theta/c^2) \end{aligned}$$

where θ is the angle between the vector $\vec{r}'$ and velocity $\vec{v}$ (the z' direction). Consequently

$$\begin{aligned} E'_x &= \frac{\gamma q x'}{r'^3}\,\frac{1}{\gamma^3(1 - v^2\sin^2\theta/c^2)^{3/2}} \\ E'_y &= \frac{\gamma q y'}{r'^3}\,\frac{1}{\gamma^3(1 - v^2\sin^2\theta/c^2)^{3/2}} \\ E'_z &= \frac{\gamma q z'}{r'^3}\,\frac{1}{\gamma^3(1 - v^2\sin^2\theta/c^2)^{3/2}} \end{aligned}$$

This expression may be written as

$$\vec{E}' = \frac{q\vec{r}'}{r'^3}\,\frac{1 - v^2/c^2}{(1 - v^2\sin^2\theta/c^2)^{3/2}}$$

The magnetic field is zero in the rest frame of the charge. However, in the moving frame there is a magnetic field. According to the transformation properties of the electromagnetic field tensor $F^{\mu\nu}$, we have

$$\begin{aligned} B'_x &= -\gamma\beta E_y = -\beta E'_y \\ B'_y &= \gamma\beta E_x = -\beta E'_x \\ B'_z &= 0 \end{aligned}$$

This magnetic field may be expressed in terms of the electric field $\vec{E}'$ as

$$\vec{B}' = \frac{\vec{v} \times \vec{E}'}{c}$$

4. A photon rocket uses light as a propellant. The initial and final rest masses of the rocket are M_i and M_f, respectively. Show that the final velocity v of the rocket, relative to its initial rest frame, is given by the equation

$$\frac{M_i}{M_f} = \sqrt{\frac{1+v/c}{1-v/c}}$$

<u>Solution</u>:
The rocket ship starts off with energy $\mathcal{E}_i = M_i c^2$, with momentum and speed zero. After some time its energy $\mathcal{E}$, momemtum p and speed v are related by the formula

$$p = \frac{\mathcal{E}}{c^2} v$$

The energy decreases, whereas the speed and momentum increase with time, because the ejected photon carry off momentum and energy. Over an infinitesimal time interval, the change in energy of the rocket and the energy $\mathcal{E}'$ carried off by the photon beam is zero:

$$d\mathcal{E} = -d\mathcal{E}'$$

The infinitesimal momentum of the ejected beam is $dp' = -d\mathcal{E}'/c$, so that the change in momentum of the rocket ship is

$$dp = d\mathcal{E}'/c$$

Thus

$$\begin{aligned}
dp &= \frac{d\mathcal{E}}{c^2}v + \frac{\mathcal{E}}{c^2}dv = -\frac{d\mathcal{E}}{c} \\
\frac{d\mathcal{E}}{c}\left(1+\frac{v}{c}\right) &= -dv\frac{\mathcal{E}}{c^2} \\
\frac{d\mathcal{E}}{\mathcal{E}} &= -\frac{dv}{c-v} \\
\ln\frac{\mathcal{E}}{\mathcal{E}_i} &= -\ln(1+v/c) \\
\mathcal{E} &= \frac{\mathcal{E}_i}{1+v/c}
\end{aligned}$$

As a consequence

$$\begin{aligned}
\mathcal{E}_f &= \frac{M_f c^2}{\sqrt{1-v^2/c^2}} = \frac{M_i c^2}{1+v/c} \\
M_f &= M_i\sqrt{\frac{1-v/c}{1+v/c}}
\end{aligned}$$

The final momentum of the rocket is

$$P_f = \frac{v\mathcal{E}_f}{c^2} = M_i c^2 \frac{v}{1+v/c}$$

The momentum of the rocket ship continually increases with time, because it is continuously ejecting photons in the backward direction.

5. Two identical particles. A and B, are moving toward one another along a common straight line with the same speed v, as measured in the lab. Show that the energy of particle A, as measured in the rest frame of particle B, is given in terms of the rest mass of A, M, as

$$Mc^2\frac{1+v^2/c^2}{1-v^2/c^2}$$

Solution:
The two identical particles are initially approaching one another with speed v. In the inertial frame in which B is at rest, A approaches it with a speed v' that is the composition of v and v. That is

$$v' = \frac{v+v}{1+(v)(v)/c^2} = \frac{2v}{1+v^2/c^2}$$

The energy of particle A in that frame is

$$\mathcal{E}_A = \frac{Mc^2}{\sqrt{1-v'^{\,2}/c^2}}$$

Now

$$\begin{aligned}
1-v'^{\,2}/c^2 &= 1-\frac{4v^2/c^2}{(1+v^2/c^2)^2} \\
&= \frac{1}{(1+v^2/c^2)^2}\left[1+2v^2/c^2+v^4/c^4-4v^2/c^2\right] \\
&= \frac{(1-v^2/c^2)^2}{(1+v^2/c^2)^2}
\end{aligned}$$

Thus

$$\mathcal{E}_A = Mc^2\frac{1+v^2/c^2}{1-v^2/c^2}$$