Physics 403: Relativity

Homework Assignment 4

Due 26 March 2007

1. Consider the three dimensional space with the line element

$$ds^2 = \frac{1}{1-r/R}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Determine the surface area of the sphere that corresponds to $r = R$.
- Determine the volume of the sphere $r \leq R$.
- Determine the distance to the center of that sphere.

Solution:
The area of the sphere at $r = R$ is determined from the angular components of the metric tensor:

$$\begin{aligned} A &= \int ds_\theta\, ds_\phi = \int_0^\pi R\, d\theta \int_0^{2\pi} R\sin\theta\, d\phi \\ &= R^2 \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = R^2 \cdot 2 \cdot 2\pi = 4\pi R^2 \end{aligned}$$

The volume of the sphere $r \leq R$ is given by

$$V = \int ds_r\, ds_\theta\, ds_\phi = 4\pi \int_0^R \frac{r^2\, dr}{\sqrt{1-r/R}}$$

In this integral we make the substitution $r = R\sin^2\rho$, where $0 \leq \rho \leq \pi/2$:

$$\begin{aligned} V &= 4\pi \int_0^{\pi/2} \frac{R^2\sin^4\rho}{\cos\rho}\, 2R\sin\rho\cos\rho\, d\rho = 8\pi R^3 \int_0^{\pi/2} \sin^5\rho\, d\rho \\ &= 8\pi R^3 \int_0^{\pi/2} \sin\rho(1-\cos^2\rho)^2\, d\rho \end{aligned}$$

We substitute $u = \cos\rho$ to obtain

$$V = 8\pi R^3 \int_0^1 du\,(1-u^2)^2 = 8\pi R^3 \frac{8}{15} = \frac{64\pi}{15}\,R^3$$

This volume is somewhat larger than the volume of a sphere in Euclidean space, $4\pi R^3/3$.
The distance to the center of the sphere is again calculated from the metric tensor:

$$D_{0R} = \int ds_r = \int_0^R \frac{dr}{\sqrt{1-r/R}} = \int_0^R \frac{\sqrt{R}\,dr}{\sqrt{R-r}} = \left[-2\sqrt{R(R-r)}\right]_{r=0}^{r=R} = 2R$$

Again, the distance s_{0R} is greater than the Euclidean distance, R.

2. The two dimensional torus can be embedded into three dimensional Euclidean space by the relations

$$\begin{aligned} x &= R\cos\theta\,[1+\rho\sin\phi] \\ y &= R\sin\theta\,[1+\rho\sin\phi] \\ z &= R\rho\cos\phi \end{aligned}$$

where $0 < \rho < 1$ is a fixed parameter. Compute the following quantities for the torus, parametrized by (θ, ϕ):

- The metric tensor induced by the embedding from the Euclidean metric.
- The Christoffel symbols
- The Riemann curvature tensor.
- The Ricci scalar

Solution:
The infinitesimal Cartesian displacements (dx, dy, dz) are calculated in terms of $(d\theta, d\phi)$ as

$$\begin{aligned} dx &= -R\sin\theta\,(1+\rho\sin\phi)\,d\theta + R\rho\cos\theta\cos\phi\,d\phi \\ dy &= R\cos\theta\,(1+\rho\sin\phi)\,d\theta + R\rho\sin\theta\cos\phi\,d\phi \\ dz &= -R\rho\sin\phi\,d\phi \end{aligned}$$

Thus we calculate the infinitesimal distance ds^2:

$$ds^2 = dx^2 + dy^2 + dz^2 = R^2\ (1+\rho\sin\phi)^2\ d\theta^2 + R^2\rho^2\ d\phi^2$$

The nonvanishing components of the metric tensor are $g_{\theta\theta} = R^2(1+\rho\sin\phi)^2$ and $g_{\phi\phi} = R^2\rho^2$.
The Christoffel symbols are given by the formula

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\ \left[\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right]$$

The following Christoffel symbols are non-vanishing:

$$\begin{aligned} \Gamma^{\phi}_{\theta\theta} &= -\frac{\cos\phi\ (1+\rho\sin\phi)}{\rho} \\ \Gamma^{\theta}_{\phi\theta} = \Gamma^{\theta}_{\theta\phi} &= \frac{\rho\cos\phi}{1+\rho\sin\phi} \end{aligned}$$

The Riemann curvature tensor is

$$R^{\alpha}_{\beta\gamma\delta} = \partial_\gamma\,\Gamma^{\alpha}_{\beta\delta} - \partial_\delta\,\Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\varepsilon}\,\Gamma^{\varepsilon}_{\beta\delta} - \Gamma^{\alpha}_{\delta\varepsilon}\,\Gamma^{\varepsilon}_{\beta\gamma}$$

The nonvanishing components of the Riemann tensor are

$$\begin{aligned} R^{\phi}_{\theta\theta\phi} &= -R^{\phi}_{\theta\phi\theta} = \frac{\sin\phi}{\rho}\ (1+\rho\cos\phi) \\ R^{\theta}_{\phi\theta\phi} &= -R^{\theta}_{\phi\phi\theta} = \frac{\rho\sin\phi}{1+\rho\sin\phi} \end{aligned}$$

The Ricci tensor is $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$. Its nonvanishing components are

$$\begin{aligned} R_{\theta\theta} &= \frac{\sin\phi}{\rho}\ (1+\rho\sin\phi) \\ R_{\phi\phi} &= \frac{\rho\sin\phi}{1+\rho\sin\phi} \end{aligned}$$

The Ricci scalar curvature is

$$R = g^{\mu\nu}\, R_{\mu\nu} = \frac{2}{R^2\rho}\frac{\sin\phi}{1+\rho\sin\phi}$$

We may compute the Killing vectors $k_\mu = [k_\theta(\theta,\phi),\ k_\phi(\theta,\phi)]$ for this geometry by solving the equations

$$\partial_\mu\, k_\nu + \partial_\nu\, k_\mu - 2\Gamma^\lambda_{\mu\nu}\, k_\lambda = 0$$

That is,

$$\begin{aligned}
\partial_\theta\, k_\theta &= \Gamma^\phi_{\theta\theta}\, k_\phi = -\frac{\cos\phi(1+\rho\sin\phi)}{\rho}\, k_\phi \\
\partial_\theta\, k_\phi + \partial_\phi\, k_\theta &= 2\,\Gamma^\theta_{\theta\phi}\, k_\theta = 2\,\frac{\rho\cos\phi}{1+\rho\sin\phi}\, k_\theta \\
\partial_\phi k_\phi &= 0
\end{aligned}$$

From the last relation we conclude that k_ϕ is independent of the variable ϕ. The second relation can be written as

$$\begin{aligned}
\partial_\phi\, k_\theta - 2\frac{\rho\cos\phi}{1+\rho\sin\phi}\, k_\theta &= -\partial_\theta\, k_\phi \\
(1+\rho\sin\phi)^2\, \partial_\phi\left[\frac{k_\theta}{(1+\rho\sin\phi)^2}\right] &= -\partial_\theta\, k_\phi
\end{aligned}$$

We define the quantity $g(\theta) = -\partial_\theta k_\phi$, and write this equation as

$$\partial_\phi\left(\frac{k_\theta}{(1+\rho\sin\phi)^2}\right) = \frac{g(\theta)}{(1+\rho\sin\phi)^2}$$

Let us integrate with respect to the variable ϕ to obtain

$$k_\theta = (1+\rho\sin\phi)^2\left[f(\theta) + g(\theta)\int_{\pi/2}^{\phi}\frac{d\phi'}{(1+\rho\sin\phi')^2}\right]$$

We insert this relation into the first Killing vector relation, to obtain

$$(1+\rho\sin\phi)^2\left[f'(\theta) + g'(\theta)\int_{\pi/2}^{\phi}\frac{d\phi'}{(1+\rho\sin\phi')^2}\right] = -\frac{\cos\phi(1+\rho\sin\phi)}{\rho}\, k_\phi$$

This equation must be satisfied at every value of the variable ϕ. Setting $\phi = \pi/2$, we establish that $f'(\theta) = 0$ at all θ. Consequently,

$$g'(\theta) \int_{\pi/2}^{\phi} \frac{d\phi'}{(1+\rho\sin\phi')^2} = -\frac{\cos\phi}{\rho(1+\rho\sin\phi)}\, k_\phi(\theta)$$

We differentiate both sides with respect to ϕ to obtain

$$\frac{g'(\theta)}{(1+\rho\sin\phi)^2} = \left[\frac{\sin\phi}{\rho(1+\rho\sin\phi)} + \frac{\cos\phi}{(1+\rho\sin\phi)^2}\right] k_\phi(\theta)$$

Since the relation must also be true at all ϕ, it follows that $k_\phi(\theta) = 0$ and $g'(\theta) = 0$. Furthermore, since $g(\theta) = -\partial_\theta\, k_\phi$, it follows that $g(\theta) = 0$. Consequently, $f(\theta) = f_0$ is a constant, and the only Killing vector is

$$k_\mu = f_0\, [(1+\rho\sin\theta)^2, 0]$$

Note that the corresponding contravariant Killing vector is

$$k^\mu = g^{\mu\lambda}\, k_\lambda = f_0\, [1, 0]$$

The existence of this Killing vector may be established from the fact that the metric tensor is independent of the coordinate θ. We have shown that it is the only Killing vector for this metric geometry.

3. Show that for any two-dimensional manifold the covariant curvature tensor has the form

$$R_{ab,cd} = \kappa\, [g_{ac}\, g_{bd} - g_{ad}\, g_{bc}]$$

where κ may be a function of the coordinates. Why does this result not generalize to manifolds of higher dimensions?
Solution:
The covariant fourth order Riemann tensor has the following symmetry properties:

$$R_{ab,cd} = -R_{ba,cd} = -R_{ab,dc}$$

In any two-dimensional space there can be only one independent component of that tensor, since the indices (a, b) as well as (c, d), must be distinct.
Thus, of the 16 components of $R_{ab,cd}$, we have

$$R_{12,21} = -R_{21,21} = -R_{12,12} = R_{21,12}$$

and all other components vanish.
The tensor

$$g_{ac}\, g_{bd} - g_{ab}\, g_{cd}$$

has the same symmetry properties as $R_{ab,cd}$, so that the two tensors must be proportional:

$$R_{ab,cd} = \kappa\, [g_{ac}g_{bd} - g_{ab}g_{cd}]$$

where κ may depend upon the two coordinates (u_1, u_2).
For the example considered in Problem 2, we have

$$\begin{aligned} R_{\phi\theta,\theta\phi} &= g_{\phi\phi}\, R^{\phi}_{\theta\theta\phi} = -R^2\rho\sin\phi\,(1+\rho\sin\phi) \\ R_{\theta\phi,\theta\phi} &= g_{\theta\theta}\, R^{\theta}_{\phi\theta\phi} = R^2\rho\sin\phi\,(1+\rho\sin\phi) \\ g_{\theta\theta}g_{\phi\phi} &= R^4\rho^2(1+\rho\sin\phi)^2 \end{aligned}$$

The structure is evident in this case, with

$$\kappa = \frac{\sin\phi}{\rho R^2\,(1+\rho\sin\phi)}$$

Note that the scalar curvature is $R = 2\kappa$.

4. Consider the Hyperbolic Plane defined by the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

where $y \geq 0$. Show that the geodesics are semi-circles centered on the x-axis or vertical lines parallel to the y-axis. Determine $x(s)$ and $y(s)$ as functions of the length s along these curves.
Solution:
The line element ds^2 is invariant under a change in the scale of coordinates; $(x,y) \to (\lambda\, x, \lambda\, y)$. The components of the metric tensor are

$$\begin{aligned} g_{xx} = 1/g^{xx} &= 1/y^2 \\ g_{yy} = 1/g^{yy} &= 1/y^2 \\ g_{xy} = g_{yx} &= 0 \end{aligned}$$

The Christoffel symbols are determined by the formula

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left[\partial_{\nu}\, g_{\rho\mu} + \partial_{\mu}\, g_{\rho\nu} - \partial_{\rho}\, g_{\mu\nu} \right]$$

The non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma^{x}_{xy} = \Gamma^{x}_{yx} &= \frac{1}{2} g^{xx}\, \partial_y g_{xx} = -\frac{1}{y} \\ \Gamma^{y}_{xx} &= -\frac{1}{2} g^{yy}\, \partial_y g_{xx} = \frac{1}{y} \\ \Gamma^{y}_{yy} &= \frac{1}{2} g^{yy}\, \partial_y g_{yy} = -\frac{1}{y} \end{aligned}$$

The Riemann tensor may be computed from the Christoffel symbols:

$$\begin{aligned} R^{x}_{yxy} &= -\partial_y \Gamma^{x}_{yx} + \Gamma^{x}_{xy}\Gamma^{y}_{yy} - \Gamma^{x}_{yx}\Gamma^{x}_{xy} = -\frac{1}{y^2} \\ R_{xyxy} &= g_{xx} R^{x}_{yxy} = -\frac{1}{y^4} \end{aligned}$$

All other components of the Riemann tensor may be obtained using the formula in Problem 3, with $\kappa = -1$. Note that the scalar curvature is $R = -2$, and that it is scale-invariant.

The equation for the geodesics $u^{\alpha}(s)$ is

$$\frac{d^2 u^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = 0$$

That is,

$$\begin{aligned} \frac{d^2 x}{ds^2} - \frac{2}{y} \frac{dx}{ds} \frac{dy}{ds} &= 0 \\ \frac{d^2 y}{ds^2} + \frac{1}{y} \left(\frac{dx}{ds} \right)^2 - \frac{1}{y} \left(\frac{dy}{ds} \right)^2 &= 0 \end{aligned}$$

It follows from the definition of the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

that

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = y^2$$

(This relation may also be obtained from the geodesic equations themselves.) We use this relation to case the geodesic equation for $y(s)$ into the form

$$y\,\frac{d^2y}{ds^2} + y^2 = 2\left(\frac{dy}{ds}\right)^2$$

Adopting the notation $y' = dy/ds$, we may write

$$\frac{d^2y}{ds^2} = \frac{dy'}{ds} = \frac{dy'}{dy}\,\frac{dy}{ds} = y'\,\frac{dy'}{dy}$$

Thus the geodesic equation for y takes the form

$$y\,y'\,\frac{dy'}{dy} + y^2 = 2y'^2$$

This is a nonlinear first order differential equation for y' as a function of y; $y'(y)$. We express it in terms of $t(y) = y'^2$:

$$\frac{y}{2}\,\frac{dt}{dy} + y^2 = 2t$$

or

$$\frac{dt}{dy} - \frac{4}{y}\,t = -2y$$

We multiply this linear first order differential equation for t by the integrating factor $1/y^4$ and solve it:

$$\begin{aligned}\frac{1}{y^4}\frac{dt}{dy}-\frac{4}{y^5}t &= -\frac{2}{y^3}\\ \frac{d}{dy}\left(\frac{t}{y^4}\right) &= \frac{d}{dy}\left(\frac{1}{y^2}\right)\\ \frac{t}{y^4} &= -\kappa^2+\frac{1}{y^2}\\ t &= y^2-\kappa^2 y^4\end{aligned}$$

Equivalently, we have

$$\begin{aligned}y'=\frac{dy}{ds} &= y\sqrt{1-\kappa^2 y^2}\\ \frac{dy}{y^2\sqrt{1/y^2-\kappa^2}} &= ds\end{aligned}$$

Let us define the variable $u(s)=1/y(s)$, so that $du=-dy/y^2$ and

$$\frac{du}{\sqrt{u^2-\kappa^2}}=-ds$$

This equation may be integrated directly to obtain $u=\kappa\cosh(s-s_0)$, or

$$y(s)=\frac{1}{\kappa\cosh(s-s_0)}$$

We may calculate $x(s)$ using this result for $y(s)$ the metric tensor relation:

$$\left(\frac{dx}{ds}\right)^2=y^2-\left(\frac{dy}{ds}\right)^2=y^2-y^2(1-\kappa^2y^2)=\kappa^2y^4$$

We take the positive square root to have

$$\frac{dx}{ds}=\kappa=\frac{1}{\kappa\cosh^2(s-s_0)}$$

Consequently,

$$x-x_0=\frac{1}{\kappa}\tanh(s-s_0)$$

We also obtain

$$(x-x_0)^2+y^2=\frac{1+\sinh^2(s-s_0)}{\kappa^2\ \cosh^2(s-s_0)}=\frac{1}{\kappa^2}$$

The geodesic trajectory is thus a circle of radius $R=1/\kappa$ centered at the point $(x_0,0)$, which lies on the x-axis. Actually, because $y(s)$ is always positive, we should regard it as a semi-circle in the upper half y-plane. In the limiting case $\kappa\longrightarrow 0$, the geodesics become half-lines, $x=x_0; y>0$.
To summarize, the geodesic trajectory is parametrized by the formulas

$$\begin{aligned} y(s) &= \frac{R}{\cosh(s-s_0)} \\ x(s) &= x_0+\frac{R\ \sinh(s-s_0)}{\cosh(s-s_0)} \end{aligned}$$

Actually, we can solve for the trajectory $y(x)$ more directly by determining the paths of minimum length

$$S=\int ds=\int_{x_1}^{x_2}\frac{\sqrt{1+y'^2}}{y}\,dx$$

Because the "effective Lagrangian"

$$L=\frac{\sqrt{1+y'^2}}{y}$$

does not involve the dependent variable x, the "Jacobi integral" is a constant of the motion:

$$\begin{aligned} y'\frac{\partial L}{\partial y'}-L &= J=-\frac{1}{R} \\ \frac{y'^2}{y\sqrt{1+y'^2}}-\frac{\sqrt{1+y'^2}}{y} &= -\frac{1}{R} \\ \frac{1}{y\sqrt{1+y'^2}} &= \frac{1}{R} \\ \sqrt{1+y'^2}=\frac{R}{y} & \\ y'=\sqrt{\frac{R^2}{y^2}-1} & \end{aligned}$$

We integrate this equation to obtain

$$\begin{aligned}\int^y \frac{y\,dy}{\sqrt{R^2-y^2}} &= \int^x dx = x-x_0\\ -\sqrt{R^2-y^2} &= x-x_0\\ (x-x_0)^2+y^2 &= R^2\end{aligned}$$

To determine the arc length of a path, we use the original metric formula:

$$\begin{aligned}\frac{ds}{dx} &= \frac{\sqrt{1+y'^2}}{y} = \frac{R}{y^2} = \frac{R}{R^2-(x-x_0)^2}\\ ds &= \frac{Rdx}{R^2-(x-x_0)^2}\end{aligned}$$

We make the substitution $x = x_0 + R\ \sin\theta$ to obtain

$$\begin{aligned}ds &= R\ \sec\theta\ d\theta\\ s-s_0 &= \ln(\sec\theta+\tan\theta)\\ e^{s-s_0} &= \sec\theta+\tan\theta\\ \sec^2\theta &= \tan^2\theta-2\ \tan\theta\ e^{s-s_0}+e^{2(s-s_0)}\\ \tan\theta &= \sinh(s-s_0)\end{aligned}$$

Note also that $\sin\theta = \tanh(s-s_0)$, so that

$$\begin{aligned}x-x_0 &= R\ \tanh(s-s_0)\\ y &= R\ \cos\theta = \frac{R}{\cosh(s-s_0)}\end{aligned}$$

5. In a four-dimensional Minkowski space with coordinates (t,x,y,z), a 3-hyperboloid is defined by $t^2-x^2-y^2-z^2 = R^2$. Show that the metric on the 3-surface of the hyperboloid can be written in the form

$$-ds^2 = R^2\ [d\chi^2+\sinh^2\chi\ (d\theta^2+\sin^2\theta\ d\phi^2)\ ]$$

Show that the total volume of the 3-hyperboloid is infinite.
Solution:
Let us define the coordinates (χ, θ, ϕ) in terms of (t,x,y,z) by

$$\begin{aligned} t &= R\cosh\chi \\ z &= R\sinh\chi\, \cos\theta \\ y &= R\sinh\chi\, \sin\theta\, \sin\phi \\ x &= R\sinh\chi\, \sin\theta\, \cos\phi \end{aligned}$$

With these coordinates, we automatically impose the constraint $t^2 - x^2 - y^2 - z^2 = R^2$. The corresponding changes in these coordinates are

$$\begin{aligned} dt &= R\sinh\chi\, d\chi \\ dz &= R\cosh\chi\cos\theta\, d\chi - R\sinh\chi\sin\theta\, d\theta \\ dy &= R\cosh\chi\sin\theta\cos\phi\, d\chi + R\sinh\chi\cos\theta\sin\phi\, d\theta - R\sinh\chi\sin\theta\cos\phi\, d\phi \\ dx &= R\cosh\chi\sin\theta\cos\phi\, d\chi + R\sinh\chi\cos\theta\cos\phi\, d\theta - R\sinh\chi\sin\theta\sin\phi\, d\phi \end{aligned}$$

The Minkowski space metric may thus be written

$$\begin{aligned} ds^2 &= dt^2 - dz^2 - dy^2 - dz^2 = -ds_\chi^2 - ds_\theta^2 - ds_\phi^2 \\ &= -R^2\, d\chi^2 - R^2\sinh^2\chi\, d\theta^2 - R^2\sinh^2\chi\sin^2\theta\, d\phi^2 \end{aligned}$$

The total three-volume of the hyperboloidal region $\chi \leq \chi_0$ is

$$\begin{aligned} V &= \int ds_\chi\, ds_\theta\, ds_\phi = \int_0^{\chi_0} R\, d\chi \int_0^{\pi} R\sinh\chi\, d\theta \int_0^{2\pi} R\sinh\chi\sin\theta\, d\phi \\ &= 4\pi R^3 \int_0^{\chi_0} \sinh^2\chi\, d\chi = 2\pi R^3 \int_0^{\chi_0} (\cosh 2\chi - 1) = \pi R^3(\sinh 2\chi_0 - 2\chi_0) \end{aligned}$$

The volume becomes infinite in the limit $\chi_0 \to \infty$.
We could determine the geodesics for this surface by obtaining and solving the geodesic equations for $(\chi(s), \theta(s), \phi(s))$. Instead, we will analyze the geodesic equations for $(t(s), x(s), y(s), z(s))$, using Lagrange multipliers to impose the constraint

$$t(s)^2 - x(s)^2 - y(s)^2 - z(s)^2 = R^2$$

The modified Lagrangian is

$$L = \left[\left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2 - \left(\frac{dz}{ds}\right)^2 \right] + \kappa^2 \left[t^2 - x^2 - y^2 - z^2\right]$$

The Euler-Lagrange equations are

$$\begin{aligned}
\frac{d^2t}{ds^2} - \kappa^2 t &= 0 \\
\frac{d^2x}{ds^2} - \kappa^2 x &= 0 \\
\frac{d^2y}{ds^2} - \kappa^2 y &= 0 \\
\frac{d^2z}{ds^2} - \kappa^2 z &= 0
\end{aligned}$$

The solution that passes through the first point on the hyperboloid, (t_1, x_1, y_1, z_1) at $s = 0$, is

$$\begin{aligned}
t(s) &= t_1 \cosh \kappa s + d \sinh \kappa s \\
x(s) &= x_1 \cosh \kappa s + a \sinh \kappa s \\
y(s) &= y_1 \cosh \kappa s + b \sinh \kappa s \\
z(s) &= z_1 \cosh \kappa s + c \sinh \kappa s
\end{aligned}$$

We the curve satisfies the constraint

$$t(s)^2 - x(s)^2 - y(s)^2 - z(s)^2 = R^2$$

under the following conditions:

$$\begin{aligned}
t_1^2 - x_1^2 - y_1^2 - z_1^2 &= R^2 \\
t_1\, d - x_1\, a - y_1\, b - z_1\, c &= 0 \\
a^2 - b^2 + c^2 - d^2 &= R^2
\end{aligned}$$

The parameter κ can be determined from the constraint

$$\left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2 - \left(\frac{dz}{ds}\right)^2 = -1$$

At $s = 0$ we obtain the relation

$$\kappa^2\,(d^2 - a^2 - b^2 - c^2) = -1$$

or $\kappa R = 1$. Under the conditions given above, this relation is satisfied everywhere on the curve.

Next we determine S, the path length on the hyperboloid between the points (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2):

$$\begin{array}{rcl} t_2 - t_1 \cosh\rho &=& d \sinh\rho \\ x_2 - x_1 \cosh\rho &=& a \sinh\rho \\ y_2 - y_1 \cosh\rho &=& b \sinh\rho \\ z_2 - z_1 \cosh\rho &=& c \sinh\rho \end{array}$$

where $\rho = \kappa S = S/R$. We square these terms and combine them to determine the parameter ρ:

$$\begin{array}{l} (t_2 - t_1 \cosh\rho)^2 - (x_2 - x_1 \cosh\rho)^2 - (y_2 - y_1 \cosh\rho)^2 - (z_2 - z_1 \cosh\rho)^2 \\ = (d \sinh\rho)^2 - (a \sinh\rho)^2 - (b \sinh\rho)^2 - (c \sinh\rho)^2\ ; \\ R^2 - 2(t_1\,t_2 - x_1\,x_2 - y_1\,y_2 - z_1\,z_2) \cosh\rho + R^2 \cosh^2\rho = -R^2 \sinh^2\rho\ ; \\ (t_1\,t_2 - x_1\,x_2 - y_1\,y_2 - z_1\,z_2) \cosh\rho = R^2 \cosh^2\rho\ ; \\ \cosh\rho = (t_1\,t_2 - x_1\,x_2 - y_1\,y_2 - z_1\,z_2)/R^2\ . \end{array}$$

One may show that $t_1\,t_2 - x_1\,x_2 - y_1 y_2 - z_1\,z_2 \geq R^2$, so that $\cosh\rho \geq 1$.

$$\begin{array}{rcl} t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 &=& \sqrt{r_1^2 + R^2}\sqrt{r_2^2 + R^2} - \vec{r}_1 \cdot \vec{r}_2 \\ &\geq& \sqrt{r_1^2 + R^2}\sqrt{r_2^2 + R^2} - r_1 r_2 \end{array}$$

Furthermore, we have

$$\begin{aligned}
(r_1 - r_2)^2 &\geq 0\,; \\
(r_1^2 + R^2)\,(r_2^2 + R^2) &\geq (r_1\,r_2 + R^2)^2\,; \\
\sqrt{r_1^2 + R^2}\,\sqrt{r_2^2 + R^2} &\geq r_1\,r_2 + r^2\,.
\end{aligned}$$

The result is established. The coefficients (a,b,c,d) can then be determined:

$$\begin{aligned}
d &= (t_2 - t_1 \cosh\rho)/\sinh\rho \\
a &= (x_2 - x_1 \cosh\rho)/\sinh\rho \\
b &= (y_2 - y_1 \cosh\rho)/\sinh\rho \\
c &= (z_2 - z_1 \cosh\rho)/\sinh\rho
\end{aligned}$$

They automatically satisfy the relation $a^2 + b^2 + c^2 - d^2 = R^2$. In addition, we may show by direct substitution that

$$t_1\,d - x_1\,a - y_1\,b - z_1\,c = \frac{1}{\sinh\rho}\,[(t_1t_2 - x_1x_2 - y_1y_2 - z_1z_2) - R^2\cosh\rho] = 0$$

as required. The geodesic curve on the hyperboloid lies on a plane passing through the origin:

$$\alpha\,x(s) + \beta\,y(s) + \gamma\,z(s) + \delta\,t(s) = 0$$

The parameters $(\alpha,\beta,\gamma,\delta)$ must satisfy the constraints

$$\begin{aligned}
\alpha\,x_1 + \beta\,y_1 + \gamma\,z_1 + \delta\,t_1 &= 0 \\
\alpha\,x_2 + \beta\,y_2 + \gamma\,z_2 + \delta\,t_2 &= 0
\end{aligned}$$

The solution for $(\alpha,\beta,\gamma,\delta)$ is not unique. In fact, there is a two-parameter family of such planes in four-dimensional Minkowski space.
This hyperboloid is a three dimensional surface in four-dimensional Minkowski space. the (Euclidean) length on this surface is

$$-ds^2 = R^2\,[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,]$$

so that

$$\begin{aligned}
g_{\chi\,\chi} &= R^2 \\
g_{\theta\,\theta} &= R^2\ \sinh^2\chi \\
g_{\phi\,\phi} &= R^2\ \sinh^2\chi\ \sin^2\theta
\end{aligned}$$

The following Christoffel symbols are non-vanishing:

$$\begin{aligned}
\Gamma^{\chi}_{\theta\,\theta} &= \frac{1}{2}g^{\chi\,\chi}\,(-\partial_\chi\,g_{\theta\,\theta}) = -\sinh\chi\ \cosh\chi \\
\Gamma^{\chi}_{\phi\,\phi} &= \frac{1}{2}g^{\chi\,\chi}\,(-\partial_\chi\,g_{\phi\,\phi}) = -\sinh\chi\ \cosh\chi\ \sin^2\theta \\
\Gamma^{\theta}_{\theta\,\chi} = \Gamma^{\theta}_{\chi\,\theta} &= \frac{1}{2}g^{\theta\,\theta}\,(\partial_\chi\,g_{\theta\,\theta}) = \frac{\cosh\chi}{\sinh\chi} \\
\Gamma^{\theta}_{\phi\,\phi} &= \frac{1}{2}g^{\theta\,\theta}\,(-\partial_\theta\,g_{\phi\,\phi}) = -\sin\theta\ \cos\theta \\
\Gamma^{\phi}_{\phi\,\chi} = \Gamma^{\phi}_{\chi\,\phi} &= \frac{1}{2}g^{\phi\,\phi}\,(\partial_\chi\,g_{\phi\,\phi}) = \frac{\cosh\chi}{\sinh\chi} \\
\Gamma^{\phi}_{\phi\,\theta} = \Gamma^{\phi}_{\theta\,\phi} &= \frac{1}{2}g^{\phi\,\phi}\,(\partial_\theta\,g_{\phi\,\phi}) = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

The non-vanishing components of R^a_{bcd} (along with their antisymmetric counterparts $R^a_{bdc} = -R^a_{bcd}$) are

$$\begin{aligned}
R^{\chi}_{\theta\,\chi\,\theta} &= \partial_\chi\,\Gamma^{\chi}_{\theta\,\theta} - \Gamma^{\chi}_{\theta\,\theta}\,\Gamma^{\theta}_{\theta\,\chi} = -\sinh^2\chi \\
R^{\chi}_{\phi\,\chi\,\phi} &= \partial_\chi\,\Gamma^{\chi}_{\phi\,\phi} - \Gamma^{\chi}_{\phi\,\phi}\Gamma^{\phi}_{\phi\,\chi} = -\sinh^2\chi\sin^2\theta \\
R^{\theta}_{\chi\,\theta\,\chi} &= \partial_\chi\,\Gamma^{\theta}_{\chi\,\theta} - \Gamma^{\theta}_{\chi\,\theta}\,\Gamma^{\theta}_{\chi\,\theta} = -1 \\
R^{\theta}_{\phi\,\theta\,\phi} &= \partial_\theta\,\Gamma^{\theta}_{\phi\,\phi} + \Gamma^{\theta}_{\theta\,\chi}\Gamma^{\chi}_{\phi\,\phi} - \Gamma^{\theta}_{\phi\,\phi}\Gamma^{\phi}_{\theta\,\phi} = -\sinh^2\theta\ \sin^2\phi \\
R^{\phi}_{\chi\,\phi\,\chi} &= -\partial_\chi\,\Gamma^{\theta}_{\chi\,\theta} - \Gamma^{\theta}_{\chi\,\theta}\Gamma^{\theta}_{\theta\,\chi} = -1 \\
R^{\phi}_{\theta\,\phi\,\theta} &= -\partial_\theta\,\Gamma^{\phi}\theta\,\phi + \Gamma^{\phi}_{\phi\,\chi}\Gamma^{\chi}_{\theta\,\theta} - \Gamma^{\phi}_{\theta\,\phi}\Gamma^{\phi}_{\theta\,\phi} = -\sinh^2\chi
\end{aligned}$$

The diagonal components of the Ricci tensor are

$$\begin{aligned}
R_{\chi\,\chi} &= -2 \\
R_{\theta\,\theta} &= -2\sinh^2\chi \\
R_{\phi\,\phi} &= -2\sinh^2\chi\ \sin^2\theta
\end{aligned}$$

The Ricci scalar is given by the simple formula $R = -2$.