Physics 403: Relativity

Homework Assignment 5

Due 09 April 2007

1. From Poisson’s equation $\nabla^2 \phi = 4\pi G \rho$, show that the gravitational potential outside a spherical object of mass $M$ at a radial distance $r$ from its center is given by $\phi(r) = -GM/r$. What is the form of $\phi(r)$ inside a uniform spherical body?

Solution:

We must find an appropriate solution of the partial differential equation

$$\nabla^2 \phi = \begin{cases} 4\pi G \rho & r < R \\ 0 & r > R \end{cases}$$

The potential function should be spherically symmetric, $\phi(r)$, so that

$$\nabla^2 \phi(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \phi(r))$$

We begin by considering the region outside the spherical object; $r > R$, so that

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r \phi) = 0$$

$$r \phi = A + Br$$

$$\phi(r) = \frac{A}{r} + B$$

Let us adopt the convention $\phi(r = \infty) = 0$, so that $B = 0$ and $\phi = A/r$. The constant $A$ is related to the total mass $M$ inside the sphere of radius $R$. To show this, we integrate over the interior of a concentric sphere of radius $r > R$:

$$\int dV \nabla^2 \phi = 4\pi G \int \rho dV$$

$$\int dV \text{div} (\text{grad} \phi) = 4\pi G M$$

$$\oint_r dS \frac{\partial \phi}{\partial r} = 4\pi G M$$

We have used the divergence theorem in obtaining the integral over the (exterior) surface of the sphere of radius $r$ in the last line. From the relation $\phi = A/r$, on the surface of the sphere we obtain
\[ \frac{\partial \phi}{\partial r} = -\frac{A}{r^2} \]

so that

\[ \oint dS \frac{\partial \phi}{\partial r} = -\frac{A}{r^2} \oint dS = -\frac{A}{r^2} (4\pi r^2) = -4\pi A = 4\pi GM \]

Thus \( A = -GM \), so that for \( r > R \) we have

\[ \phi(r) = -\frac{GM}{r} \]

Note that the total mass \( M \) inside the sphere is given in terms of its density \( \rho \) by

\[ M = \frac{4\pi}{3} R^3 \rho \]

Next we consider the potential inside the sphere; \( r < R \):

\[ \nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho = \frac{3GM}{R^3} \]

\[ \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = \frac{3GM}{R^3} r \]

\[ r\phi = \frac{GM}{2R^3} r^3 + C + Dr \]

\[ \phi = \frac{GM}{2R^3} r^2 + \frac{C}{r} + D \]

We must match both \( \phi(r) \) and \( \phi'(r) = \partial \phi/\partial r \) at the surface \( r = R \). We obtain

\[ \frac{GM}{2R} + \frac{C}{R} + D = \phi(R) = -\frac{GM}{R} \]

\[ \frac{GM}{R^2} - \frac{C}{R^2} = \phi'(R) = \frac{GM}{R^2} \]

Thus, \( C = 0 \) and \( D = -3GM/(2R) \). The potential inside the sphere is

\[ \phi(r) = -\frac{GM}{R} \frac{3R^2 - r^2}{2R^2} \]
The potential at \( r = 0 \), the center of the sphere, is \( \phi(0) = -3GM/(2R) \).

2. A spacetime has the metric
\[
ds^2 = d(ct)^2 - a^2(ct) \left( dx^2 + dy^2 + dz^2 \right)
\]

Show that the only non-zero Christoffel symbols are
\[
\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a \dot{a} \\
\Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \frac{\dot{a}}{a}
\]

Deduce that particles may be at rest in such a spacetime, and that for such particles the coordinate \( t \) is their proper time. Show further that the nonzero components of the Ricci tensor are
\[
R_{00} = -3 \frac{\ddot{a}}{a} \\
R_{11} = R_{22} = R_{33} = a \ddot{a} + 2 \dot{a}^2
\]

Hence show that the scalar curvature is
\[
R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)
\]

**Solution:**
Let us write the line element in terms of \((x^0, x^i, x^2, x^3)\):
\[
ds^2 = d(x^0)^2 - a^2(x^0) \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right]
\]

The (non-vanishing) diagonal components of the metric tensor are
\[
g_{00} = 1/g^{00} = 1 \\
g_{11} = 1/g^{11} = -a^2(x^0) \\
g_{22} = 1/g^{22} = -a^2(x^0) \\
g_{33} = 1/g^{33} = -a^2(x^0)
\]

For the Christoffel symbols (along with symmetric partners) we obtain
\[
\Gamma^0_{11} = \frac{1}{2} g^{00} [-\partial_0 g_{11}] = -\frac{1}{2} (-\partial_0 a^2) = a \dot{a}
\]

\[
\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a \dot{a}
\]

\[
\Gamma^1_{01} = \frac{1}{2} g^{11} (\partial_0 g_{11}) = -\frac{1}{2 a^2} (-\partial_0 a^2) = \frac{\dot{a}}{a}
\]

\[
\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}
\]

Let us calculate the components of the Riemann tensor

\[
R^l_{010} = -\partial_0 \Gamma^l_{01} - \Gamma^l_{01} \Gamma^0_{01} = -\partial_0 \left( \frac{a}{\dot{a}} \right) - \left( \frac{\dot{a}}{a} \right)^2 = -\frac{\ddot{a}}{a}
\]

\[
R^1_{010} = R^2_{020} = R^3_{030} = -\frac{\ddot{a}}{a}
\]

\[
R^0_{101} = \partial_0 \Gamma^0_{11} + \Gamma^0_{11} \Gamma^1_{10} = \partial_0 (a \ddot{a}) + (a \dot{a}) \frac{\ddot{a}}{a} = a \ddot{a} + 2 \dot{a}^2
\]

\[
R^0_{101} = R^0_{202} = R^0_{303} = a \ddot{a} + 2 \dot{a}^2
\]

The non-vanishing components of the Ricci tensor are

\[
R_{00} = R^1_{010} + R^2_{020} + R^3_{030} = -3 \frac{\ddot{a}}{a}
\]

\[
R_{11} = R^0_{101} = a \ddot{a} + 2 \dot{a}^2
\]

\[
R_{22} = R^0_{202} = a \ddot{a} + 2 \dot{a}^2
\]

\[
R_{33} = R^0_{303} = a \ddot{a} + 2 \dot{a}^2
\]
The scalar curvature is

\[ R = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \]
\[ = -3 \frac{\ddot{a}}{a} - 3 \left( a \dddot{a} + 2 \dot{a}^2 \right) = -6 \frac{\dddot{a}}{a} - 6 \left( \frac{\dot{a}}{a} \right)^2 \]

Particles move along “geodesic line” trajectories that satisfy the geodesic equation:

\[ \frac{d^2 u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0 \]

Is the trajectory for a particle at rest, \( u^\alpha = (x^0 = s, x^a = x^a_0) \) a solution of the geodesic equation? That is, are the following equations satisfied?

\[ \frac{d^2 x^0}{ds^2} + \Gamma^0_{bb} \frac{dx^b}{ds} \frac{dx^b}{ds} = 0 \]
\[ \frac{d^2 x^a}{ds^2} + \Gamma^a_{0b} \frac{dx^0}{ds} \frac{dx^b}{ds} = 0 \]

Putting the condition \( \frac{dx^b}{ds} = 0 \) into these equations, we obtain

\[ \frac{d^2 x^0}{ds^2} = 0 \]
\[ \frac{d^2 x^a}{ds^2} = 0 \]

We obtain the solution

\[ x^0 = x^2_0 + b \ s \]
\[ x^a = x^a_0 \]

Thus, a particle may remain at rest at any location in this metric.

3. In Schwarzschild geometry, we introduce the new coordinates

\[ x = r \ \sin \theta \ \cos \phi \]
\[ y = r \ \sin \theta \ \sin \phi \]
\[ z = r \ \cos \theta \]
Find the form of the line element in these coordinates.

**Solution:**

The Euclidean line element may be written in either Cartesian coordinates or spherical polar coordinates:

\[ dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \]

Equivalently, we obtain

\[ r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) = dx^2 + dy^2 + dz^2 - dr^2 \]

Furthermore, it follows from the relation for \( r(x, y, z) \), \( r^2 = x^2 + y^2 + z^2 \), that

\[ r \, dr = x \, dx + y \, dy + z \, dz \]

or

\[ dr^2 = \frac{(x \, dx + y \, dy + z \, dz)^2}{r^2} \]

Thus, we may express the Schwarzschild line element as

\[ ds^2 = (1 - \frac{2R}{r})(cdt)^2 - \frac{d\tau^2}{1 - \frac{2R}{r}} - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \]

\[ = (1 - \frac{2R}{r})(cdt)^2 - dr^2 \left[ \frac{1}{1 - \frac{2R}{r}} - 1 \right] - dx^2 - dy^2 - dz^2 \]

\[ = (1 - \frac{2R}{r})(cdt)^2 - dr^2 \frac{2R}{r - 2R} - dx^2 - dy^2 - dz^2 \]

\[ = (1 - \frac{2R}{r})(cdt)^2 - \frac{(x \, dx + y \, dy + z \, dz)^2}{r^2} \frac{2R}{r - 2R} - dx^2 - dy^2 - dz^2 \]

We write out the term

\[ (x \, dx + y \, dy + z \, dz)^2 = x^2dx^2 + y^2dy^2 + z^2dz^2 + 2xz \, dxdz + 2xy \, dx dy + 2yz \, dy dz \]

and obtain the following components of the metric tensor, expressed in terms of \( (x, y, z) \) and \( r = \sqrt{x^2 + y^2 + z^2} \):
\[
\begin{align*}
    g_{00} &= \left[ 1 - \frac{2R}{r} \right] \\
    g_{0a} = g_{a0} &= 0 \\
    g_{xx} &= -\left[ \frac{2R}{r-2R} \frac{x^2}{r^2} + 1 \right] \\
    g_{yy} &= -\left[ \frac{2R}{r-2R} \frac{y^2}{r^2} + 1 \right] \\
    g_{zz} &= -\left[ \frac{2R}{r-2R} \frac{z^2}{r^2} + 1 \right] \\
    g_{xy} = g_{yx} &= -\frac{2R}{r-2R} \frac{xy}{r^2} \\
    g_{xz} = g_{zx} &= -\frac{2R}{r-2R} \frac{xz}{r^2} \\
    g_{yz} = g_{zy} &= -\frac{2R}{r-2R} \frac{yz}{r^2}
\end{align*}
\]

Note that the Schwarzschild metric is considerably more complicated in Cartesian spatial coordinates than for polar coordinates.

4. Consider a spacetime with metric
\[
ds^2 = e^{-2ax} \, dt^2 - dx^2 - dy^2 - dz^2
\]
where the parameter \(a\) is constant.

- Find all the Christoffel symbols
- Find the geodesic equations for \(x(t)\), and show that for instantaneous zero velocity, the \(x\) component varies with uniform acceleration \(a\), as though the particle were in a uniform gravitational field.
- Determine the components of the Riemann tensor \(R_{001}^1, R_{110}^0, R_{010}^1, R_{101}^0\), and show that the rest are zero.
- From the Einstein field equations find the diagonal elements of the energy momentum tensor. Is this tensor physically acceptable?
Solution:
The nonvanishing components of the metric tensor are

\[ g_{tt} = 1 / g^{tt} = e^{-2a x} \]
\[ g_{xx} = g^{xx} = -1 \]
\[ g_{yy} = g^{yy} = -1 \]
\[ g_{zz} = g^{zz} = -1 \]

The following Christoffel (along with symmetric partners) are non-zero:

\[ \Gamma^x_{tt} = \frac{1}{2} g^{xx} \left( -\partial_x g_{tt} \right) = -\frac{1}{2} \left[ 2a e^{-2ax} \right] = -a e^{-2ax} \]
\[ \Gamma^t_{xt} = \frac{1}{2} g^{tt} \left[ \partial_x g_{tt} \right] = \frac{1}{2} e^2 a x \left[ -2a e^{-2ax} \right] = -a \]

In turn, these components of the Riemann tensor (along with antisymmetric partners) are non-vanishing:

\[ R^t_{txx} = -\Gamma^t_{xt} \Gamma^x_{tx} = -a^2 \]
\[ R^x_{ttx} = \partial_x \Gamma^x_{tt} - \Gamma^x_{tt} \Gamma^t_{xt} = 2a^2 e^{-2ax} - a^2 e^{-2ax} = a^2 e^{-2ax} \]

The following components of the Ricci tensor are non-zero:

\[ R_{xx} = R^t_{txx} = -a^2 \]
\[ R_{tt} = R^x_{ttx} = a^2 e^{-2ax} \]

Thus we may compute the scalar curvature:

\[ R = g^{xx} R_{xx} + g^{tt} R_{tt} = 2a^2 \]

The Einstein field equations have the form

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \]
Specifically, we obtain

$$
\kappa T_{tt} = R_{tt} - \frac{1}{2} g_{tt} R = a^2 e^{-2 ax} - \frac{1}{2} e^{-2 ax} 2a^2 = 0
$$

$$
\kappa T_{xx} = R_{xx} - \frac{1}{2} g_{xx} R = -a^2 - \frac{1}{2} (-1)(2a^2) = 0
$$

As a consequence, all components of the energy-momentum tensor must be zero. The geodesics must satisfy the equations

$$
\frac{d^2 \alpha^a}{ds^2} + \Gamma^a_{bc} \frac{d\alpha^b}{ds} \frac{d\alpha^c}{ds} = 0
$$

That is,

$$
\frac{d^2 t}{ds^2} = \frac{2a}{ds} \frac{dt}{ds} \frac{dx}{ds}
$$

$$
\frac{d^2 x}{ds^2} = ae^{-2ax} \left( \frac{dt}{ds} \right)^2
$$

The original metric relation is a consequence of these geodesic equations:

$$
1 = e^{-2ax} \left( \frac{dt}{ds} \right)^2 - \left( \frac{dx}{ds} \right)^2
$$

Thus the second geodesic equation may be written

$$
\frac{d^2 x}{ds^2} = a \left[ 1 + \left( \frac{dx}{ds} \right)^2 \right]
$$

Letting \( v = dx/ds \), we obtain

$$
\frac{dv}{dx} = a(1 + v^2)
$$

$$
\log(1 + v^2) = 2a (x - x_0)
$$

$$
1 + v^2 = e^2a (x-x_0)
$$
If the particle starts from rest at \( x = x_0 \), its proper speed \( v \) over short distances satisfies the relation corresponding to uniform acceleration:

\[
v^2 = 2a(x - x_0)
\]

5. Determine the surface area of a sphere in \( n \)-dimensional Euclidean space by carrying out the following steps:

- The surface area of an \( n \)-dimensional sphere of radius \( r \) is \( S_n(r) = a_n r^{n-1} \), where \( a_n \) is independent of the radius \( r \). Why?

- Show that

\[
J = \int_{-\infty}^{\infty} dx \ e^{-ax^2} = \sqrt{\frac{\pi}{a}}
\]

- Express \( J^n \) as an integral over Cartesian coordinates \( (x_1, x_2, \ldots, x_n) \) in \( n \)-dimensional Euclidean space.

- Express \( J^n \) as an integral over the coordinate \( r \), where \( r^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \).

- Evaluate that integral over \( r \), and thus determine the factor \( a_n \) in the surface area of an \( n \)-sphere.

**Solution:**

Let us begin by writing \( J^2 \) as a two-dimensional integral:

\[
J^2 = \int_{-\infty}^{\infty} dx \ e^{-ax^2} \int_{-\infty}^{\infty} dy \ e^{-ay^2}
\]

We transform this two-dimensional integral in polar coordinates:

\[
J^2 = \int_{0}^{\infty} r \ e^{-ar^2} \ d r \int_{0}^{2\pi} d \theta = \frac{\pi}{a} \int_{0}^{\infty} d \left[ -e^{-ar^2} \right] = \frac{\pi}{a}
\]

Taking the positive square root, we determine \( J = \sqrt{\pi/a} \). Next we express \( J^n \) as an integral over \( n \)-dimensional space, which we transform to polar coordinates, defining

\[
r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]
We obtain

\[ J^n = \int_0^\infty e^{-ar^2} S_n(r) \, dr \]

where \( S_n(r) \) is the area of an \( n \)-dimensional spherical hypersurface of radius \( r \).

The volume of an \( n \)-dimensional sphere of radius \( R \) is given by this integral over Cartesian coordinates:

\[ V_n(R) = \int_{x_1^2 + x_2^2 + \ldots + x_n^2 \leq R^2} dx_1 dx_2 \ldots dx_n \]

Let us express this volume as an integral over scaled coordinates \( t_i = x_i/R \):

\[ V_n(R) = R^n \int_{t_1^2 + t_2^2 + \ldots + t_n^2 \leq R^2} dt_1 dt_2 \ldots dt_n \equiv v_n R^n \]

since the integration over \( t_i \) is independent of \( R \). We may write this volume integral as

\[ V_n(R) = \int_0^R dr S_n(r) \]

so that

\[ S_n(r) = s_n r^{n-1} = n v_n r^{n-1} \]

Consequently,

\[ J^n = \left[ \frac{\pi \, a^{n/2}}{a} \right] = s_n \int_0^\infty e^{-ar^2} r^{n-1} \, dr \]

We define the variable \( u = ar^2 \) to transform that integral:

\[ J^n = \frac{s_n}{2a^{n/2}} \int u^{n/2-1} e^{-u} du \]

Using the definition of the Euler Gamma function,

\[ \Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \, du \]

we obtain the result
\[ f_n = \left[ \frac{\pi}{a} \right]^{n/2} = \frac{s_n}{2a^{n/2}} \Gamma(n/2) \]

We may thus solve for \( s_n \):

\[ s_n = 2 \frac{\pi^{n/2}}{\Gamma(n/2)} \]

The Euler Gamma function obeys the relation \( \Gamma(z + 1) = z\Gamma(z) \), as we show by integrating by parts:

\[ \Gamma(z + 1) = \int_0^\infty u^z e^{-u} \, du = \int_0^\infty u^z \, d[-e^{-u}] = [-u^z e^{-u}]_0^\infty + z \int_0^\infty u^{z-1} e^{-u} \, du = z \Gamma(z) \]

From the formulas \( \Gamma(1/2) = \sqrt{\pi} \) and \( \Gamma(1) = 1 \), we may calculate the Gamma function and \( s_n \) at any value of \( n \). Here are the values up to \( n = 12 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s_n ) numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
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</tr>
<tr>
<td>3</td>
<td>4\pi</td>
</tr>
<tr>
<td>4</td>
<td>2\pi^2</td>
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<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>\pi^3</td>
</tr>
<tr>
<td>7</td>
<td>\frac{16}{15}\pi^3</td>
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<tr>
<td>8</td>
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<td>\frac{64}{255}\pi^5</td>
</tr>
<tr>
<td>12</td>
<td>\frac{8}{60}\pi^6</td>
</tr>
</tbody>
</table>

The largest coefficient occurs at \( n = 7 \).

6. In a Schwarzschild metric, what is the volume contained in the spherical shell between Schwarzschild coordinates \( r_1 \) and \( r_2 \), where \( r_2 > r_1 > R \), where \( R \) is the Schwarzschild radius?

**Solution:**

The Schwarzschild metric is
\[ ds^2 = ds_r^2 - ds_t^2 - ds_\theta^2 - ds_\phi^2 = (1 - \frac{2R}{r})(cdt)^2 - \frac{dr^2}{1-2R/r} - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \]

The volume of the region between radius \( r = 2R \) and \( r = r' \) is

\[ V(r') = \int ds_r \int ds_\theta \int ds_\phi = \int_{2R}^{r'} \frac{dr}{\sqrt{1-2R/r}} \int_0^{\pi} r \, d\theta \int_0^{2\pi} r \sin \theta \, d\phi \]

\[ = 4\pi \int_{2R}^{r'} \frac{r^2 \, dr}{\sqrt{1-2R/r}} \]

Let us define the variable \( u = 2R/r \), so that \( dr = -2Rdu/u^2 \) and

\[ V(r') = 32\pi R^3 \int_0^{1} \frac{du}{u^4} \frac{1}{\sqrt{1-u}} \]

where the upper limit is \( u' = 2R/r' \). We define the angular variable \( \rho \) by the relation \( u = \cos^2 \rho \), so that \( du = -2\cos \rho \sin \rho \, d\rho \) and

\[ V = 64\pi R^3 \int_0^{\rho'} \sec^7 \rho \, d\rho \]

where \( \cos^2 \rho' = u' = 2R/r' \).

The integral may be evaluated in closed form \(^1\)

\[ \int_0^{\rho'} \sec^7 \rho \, d\rho = \tan \rho' \sec \rho' \left[ \frac{1}{6} \sec^4 \rho' + \frac{5}{24} \sec^2 \rho' + \frac{5}{16} \right] + \frac{5}{16} \ln [\tan \rho' + \sec \rho'] \]

Consequently, the volume \( V(r') \) is

\[ V(r') = 4\pi \left[ \sqrt{r'(r' - 2R)} \left( \frac{1}{3} r'^2 + \frac{5}{6} r'R + \frac{5}{2} R^2 \right) + \frac{5}{16} \ln \left[ \sqrt{r'/2R} + \sqrt{r' - 2R/2R} \right] \right] \]

In the limit \( r' \gg 2R \), this volume approaches the “flat space” limit, \( V(r') = 4\pi r'^3 / 3 \). The volume between radii \( r_2 \) and \( r_1 \) is \( V(r_2) - V(r_1) \).

\(^1\) Gradshteyn and Ryzhik, Table of Integrals, Series, and Products (1965), 2.526.15, p. 137