Physics 403: Relativity

Homework Assignment 5

Due 09 April 2007

1. From Poisson's equation $\nabla^2\phi = 4\pi\, G\, \rho$, show that the gravitational potential outisde a spherical object of mass M at a radial distance r from its center is given by $\phi(r) = -GM/r$. What is the form of $\phi(r)$ inside a uniform spherical body?
Solution:
We must find an appropriate solution of the partial differential equation

$$\nabla^2\phi = \begin{bmatrix} 4\pi\, G\, \rho & r < R \\ 0 & r > R \end{bmatrix}$$

The potential function should be spherically symmetric, $\phi(r)$, so that

$$\nabla^2\phi(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\phi)$$

We begin by considering the region outside the spherical object; $r > R$, so that

$$\begin{aligned} \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\phi) &= 0 \\ r\,\phi &= A + B\,r \\ \phi(r) &= \frac{A}{r} + B \end{aligned}$$

Let us adopt the convention $\phi(r=\infty) = 0$, so that $B = 0$ and $\phi = A/r$. The constant A is related to the total mass M inside the sphere of radius R. To show this, we integrate over the interior of a concentric sphere of radius $r > R$:

$$\begin{aligned} \int dV\ \nabla^2\phi &= 4\pi\, G \int \rho\, dV \\ \int dV \operatorname{div}(\operatorname{grad}\phi) &= 4\pi\, G\, M \\ \oint_r dS\, \frac{\partial\phi}{\partial r} &= 4\pi\, G\, M \end{aligned}$$

We have used the divergence theorem in obtaining the integral over the (exterior) surface of the sphere of radius r in the last line. From the relation $\phi = A/r$, on the surface of the sphere we obtain

$$\frac{\partial \phi}{\partial r} = -\frac{A}{r^2}$$

so that

$$\oint_r dS\, \frac{\partial \phi}{\partial r} = -\frac{A}{r^2} \oint_r dS = -\frac{A}{r^2}\,(4\pi\, r^2) = -4\pi\, A = 4\pi\, G\, M$$

Thus $A = -G\, M$, so that for $r > R$ we have

$$\phi(r) = -\frac{G\, M}{r}$$

Note that the total mass M inside the sphere is given in terms of its density ρ by

$$M = \frac{4\pi}{3} R^3\, \rho$$

Next we consider the potential inside the sphere; $r < R$:

$$\begin{aligned}
\nabla^2 \phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\phi) &= 4\pi\, G\, \rho = \frac{3\, G\, M}{R^3} \\
\frac{\partial^2}{\partial r^2}(r\,\phi) &= \frac{3\, G\, M}{R^3}\, r \\
r\phi &= \frac{G\, M}{2R^3}\, r^3 + C + D\, r \\
\phi &= \frac{G\, M}{2R^3}\, r^2 + \frac{C}{r} + D
\end{aligned}$$

We must match both $\phi(r)$ and $\phi'(r) = \partial\phi/\partial r$ at the surface $r = R$. We obtain

$$\begin{aligned}
\frac{G\, M}{2R} + \frac{C}{R} + D &= \phi(R) = & -\frac{G\, M}{R} \\
\frac{G\, M}{R^2} - \frac{C}{R^2} &= \phi'(R) = & \frac{G\, M}{R^2}
\end{aligned}$$

Thus, $C = 0$ and $D = -3GM/(2R)$. The potential inside the sphere is

$$\phi(r) = -\frac{G\, M}{R}\, \frac{3R^2 - r^2}{2R^2}$$

The potential at $r = 0$, the center of the sphere, is $\phi(0) = -3GM/(2R)$.

2. A spacetime has the metric

$$ds^2 = d(ct)^2 - a^2(ct)\,(dx^2 + dy^2 + dz^2)$$

Show that the only non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} &= a\,\dot{a} \\ \Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} &= \frac{\dot{a}}{a} \end{aligned}$$

Deduce that particles may be at rest in such a spacetime, and that for such particles the coordinate t is their proper time. Show further that the nonzero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} = R_{22} = R_{33} &= a\,\ddot{a} + 2\,\dot{a}^2 \end{aligned}$$

Hence show that the scalar curvature is

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$

<u>Solution</u>:
Let us write the line element in terms of (x^0, x^1, x^2, x^3):

$$ds^2 = d(x^0)^2 - a^2(x^0)\,[\,(dx^1)^2 + (dx^2)^2 + (dx^3)^2\,]$$

The (non-vanishing) diagonal components of the metric tensor are

$$\begin{aligned} g_{00} = 1/g^{00} &= 1 \\ g_{11} = 1/g^{11} &= -a^2(x^0) \\ g_{22} = 1/g^{22} &= -a^2(x^0) \\ g_{33} = 1/g^{33} &= -a^2(x^0) \end{aligned}$$

For the Christoffel symbols (along with symmetric partners) we obtain

$$
\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2} g^{00} \left[-\partial_0 \, g_{11}\right] \\
&= -\frac{1}{2}(-\partial_0 \, a^2) = a \, \dot{a} \\
\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} &= a \, \dot{a}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{01} &= \frac{1}{2} g^{11} \left(\partial_0 \, g_{11}\right) \\
&= -\frac{1}{2a^2} \left(-\partial_0 \, a^2\right) = \frac{\dot{a}}{a} \\
\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} &= \frac{\dot{a}}{a}
\end{aligned}
$$

Let us calculate the components of the Riemann tensor

$$
\begin{aligned}
R^1_{010} &= -\partial_0 \, \Gamma^1_{01} - \Gamma^1_{01} \, \Gamma^1_{01} \\
&= -\partial_0 \, \left(\frac{a}{\dot{a}}\right) - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{\ddot{a}}{a} \\
R^1_{010} = R^2_{020} = R^3_{030} &= -\frac{\ddot{a}}{a}
\end{aligned}
$$

$$
\begin{aligned}
R^0_{101} &= \partial_0 \, \Gamma^0_{11} + \Gamma^0_{11} \, \Gamma^1_{10} \\
&= \partial_0 \, (a\dot{a}) + (a \, \dot{a}) \, \frac{\dot{a}}{a} = a \, \ddot{a} + 2\dot{a}^2 \\
R^0_{101} = R^0_{202} = R^0_{303} &= a \, \ddot{a} + 2\dot{a}^2
\end{aligned}
$$

The non-vanishing components of the Ricci tensor are

$$
\begin{aligned}
R_{00} = R^1_{010} + R^2_{020} + R^3_{030} &= -3\frac{\ddot{a}}{a} \\
R_{11} = R^0_{101} &= a \, \ddot{a} + 2\dot{a}^2 \\
R_{22} = R^0_{202} &= a \, \ddot{a} + 2\dot{a}^2 \\
R_{33} = R^0_{303} &= a \, \ddot{a} + 2\dot{a}^2
\end{aligned}
$$

The scalar curvature is

$$
\begin{aligned}
R &= g^{00}\,R_{00} + g^{11}\,R_{11} + g^{22}\,R_{22} + g^{33}\,R_{33} \\
&= -3\frac{\ddot{a}}{a} - \frac{3}{a^2}(a\,\ddot{a} + 2\dot{a}^2) = -6\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right)^2
\end{aligned}
$$

Particles move along "geodesic line" trajectories that satisfy the geodesic eauation:

$$
\frac{d^2u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\,\frac{du^\beta}{ds}\,\frac{du^\gamma}{ds} = 0
$$

Is the trajectory for a particle at rest, $u^\alpha = (x^0 = s, x^a = x_0^a)$ a solution of the geodesic equation? That is, are the following equations satisfied?

$$
\begin{aligned}
\frac{d^2x^0}{ds^2} + \Gamma^0_{bb}\,\frac{dx^b}{ds}\,\frac{dx^b}{ds} &= 0 \\
\frac{d^2x^a}{ds^2} + \Gamma^a_{0a}\,\frac{dx^0}{ds}\,\frac{dx^a}{ds} &= 0
\end{aligned}
$$

Putting the condition $\frac{dx^b}{ds} = 0$ into these equations, we obtain

$$
\begin{aligned}
\frac{d^2x^0}{ds^2} &= 0 \\
\frac{d^2x^a}{ds^2} &= 0
\end{aligned}
$$

We obtain the solution

$$
\begin{aligned}
x^0 &= x_0^2 + b\,s \\
x^a &= x_0^a
\end{aligned}
$$

Thus, a particle may remain at rest at any location in this metric.

3. In Schwarzschild geometry, we introduce the new coordinates

$$
\begin{aligned}
x &= r\,\sin\theta\,\cos\phi \\
y &= r\,\sin\theta\,\sin\phi \\
z &= r\,\cos\theta
\end{aligned}
$$

Find the form of the line element in these coordinates.
Solution:
The Euclidean line element may be written in either Cartesian coordinates or spherical polar coordinates:

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Equivalently, we obtain

$$r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = dx^2 + dy^2 + dz^2 - dr^2$$

Furthermore, it follows from the relation for $r(x,y,z)$, $r^2 = x^2 + y^2 + z^2$, that

$$rdr = xdx + ydy + zdz$$

or

$$dr^2 = \frac{(xdx + ydy + zdz)^2}{r^2}$$

Thus, we may express the Schwarzschild line element as

$$\begin{aligned} ds^2 &= (1 - \frac{2R}{r})(cdt)^2 - \frac{dr^2}{1 - 2R/r} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &= (1 - \frac{2R}{r})(cdt)^2 - dr^2\left[\frac{1}{1 - 2R/r} - 1\right] - dx^2 - dy^2 - dz^2 \\ &= (1 - \frac{2R}{r})(cdt)^2 - dr^2 \frac{2R}{r - 2R} - dx^2 - dy^2 - dz^2 \\ &= (1 - \frac{2R}{r})(cdt)^2 - \frac{(xdx + ydy + zdz)^2}{r^2}\ \frac{2R}{r - 2R} - dx^2 - dy^2 - dz^2 \end{aligned}$$

We write out the term

$$(xdx + ydy + zdz)^2 = x^2dx^2 + y^2dy^2 + z^2dz^2 + 2xz\, dxdz + 2xy\, dxdy + 2yz\, dydz$$

and obtain the following components of the metric tensor, expressed in terms of (x,y,z) and $r = \sqrt{x^2 + y^2 + z^2}$:

$$
\begin{aligned}
g_{00} &= \left[1-\frac{2R}{r}\right] \\
g_{0a}=g_{a0} &= 0 \\
g_{xx} &= -\left[\frac{2R}{r-2R}\frac{x^2}{r^2}+1\right] \\
g_{yy} &= -\left[\frac{2R}{r-2R}\frac{y^2}{r^2}+1\right] \\
g_{zz} &= -\left[\frac{2R}{r-2R}\frac{z^2}{r^2}+1\right] \\
g_{xy}=g_{yx} &= -\frac{2R}{r-2R}\frac{xy}{r^2} \\
g_{xz}=g_{zx} &= -\frac{2R}{r-2R}\frac{xz}{r^2} \\
g_{yz}=g_{zy} &= -\frac{2R}{r-2R}\frac{yz}{r^2}
\end{aligned}
$$

Note that the Schwarzschild metric is considerably more complicated in Cartesian spatial coordinates than for polar coordinates.

4. Consider a spacetime with metric

$$ds^2 = e^{-2ax}\, dt^2 - dx^2 - dy^2 - dz^2$$

where the parameter a is constant.

- Find all the Christoffel symbols
- Find the geodesic equations for $x(t)$, and show that for instantaneous zero velocity, the x component varies with uniform acceleration a, as though the particle were in a uniform gravitational field.
- Determine the components of the Riemann tensor $R^1_{001}, R^0_{110}, R^1_{010}, R^0_{101}$, and show that the rest are zero.
- From the Einstein field equations find the diagonal elements of the energy momentum tensor. Is this tensor physically acceptable?

Solution:
The nonvanishing components of the metric tensor are

$$\begin{aligned} g_{tt} = 1/g^{tt} &= e^{-2\,a\,x} \\ g_{xx} = g^{xx} &= -1 \\ g_{yy} = g^{yy} &= -1 \\ g_{zz} = g^{zz} &= -1 \end{aligned}$$

The following Christoffel (along with symmetric partners) are non-zero:

$$\begin{aligned} \Gamma^x_{tt} &= \frac{1}{2} g^{xx} \left[-\partial_x \, g_{tt} \right] = -\frac{1}{2} \left[2\,a\,e^{-2\,a\,x} \right] = -a\,e^{-2\,a\,x} \\ \Gamma^t_{xt} &= \frac{1}{2} g^{tt} \left[\partial_x \, g_{tt} \right] = \frac{1}{2}\, e^{2\,a\,x} \left[-\, 2a\, e^{-2\,a\,x} \right] = -a \end{aligned}$$

In turn, these components of the Riemann tensor (along with antisymmetric partners) are non-vanishing:

$$\begin{aligned} R^t_{xtx} &= -\Gamma^t_{xt}\,\Gamma^t_{tx} = -a^2 \\ R^x_{txt} &= \partial_x\,\Gamma^x_{tt} - \Gamma^x_{tt}\,\Gamma^t_{xt} = 2a^2\,e^{-2\,a\,x} - a^2\,e^{-2\,a\,x} = a^2\,e^{-2\,a\,x} \end{aligned}$$

The following components of the Ricci tensor are non-zero:

$$\begin{aligned} R_{xx} &= R^t_{xtx} = -a^2 \\ R_{tt} &= R^x_{txt} = a^2 e^{-2\,a\,x} \end{aligned}$$

Thus we may compute the scalar curvature:

$$R = g^{xx} R_{xx} + g^{tt} R_{tt} = 2a^2$$

The Einstein field equations have the form

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \kappa\, T_{\mu\nu}$$

Specifically, we obtain

$$
\begin{aligned}
\kappa\, T_{tt} &= R_{tt} - \frac{1}{2}\, g_{tt}\, R \\
&= a^2\, e^{-2\,a\,x} - \frac{1}{2} e^{-2\,a\,x} 2a^2 = 0 \\
\kappa\, T_{xx} &= R_{xx} - \frac{1}{2}\, g_{xx}\, R \\
&= -a^2 - \frac{1}{2}(-1)(2\,a^2) = 0
\end{aligned}
$$

As a consequence,all components of the energy-momentum tensor must be zero. The geodesics must satisfy the equations

$$\frac{d^2u^a}{ds^2} + \Gamma^a_{bc} \frac{du^b}{ds}\frac{du^c}{ds} = 0$$

That is,

$$
\begin{aligned}
\frac{d^2t}{ds^2} &= 2a \frac{dt}{ds}\frac{dx}{ds} \\
\frac{d^2x}{ds^2} &= a e^{-2\,a\,x} \left(\frac{dt}{ds}\right)^2
\end{aligned}
$$

The original metric relation is a consequence of these geodesic equations:

$$1 = e^{-2\,a\,x} \left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2$$

Thus the second geodesic equation may be written

$$\frac{d^2x}{ds^2} = a \left[1 + \left(\frac{dx}{ds}\right)^2\right]$$

Letting $v = dx/ds$, we obtain

$$
\begin{aligned}
v\frac{dv}{dx} &= a(1+v^2) \\
\log(1+v^2) &= 2\,a\,(x - x_0) \\
1 + v^2 &= e^{2\,a\,(x-x_0)}
\end{aligned}
$$

If the particle starts from rest at $x = x_0$, its proper speed v over short distances satisfies the relation corresponding to uniform acceleration:

$$v^2 = 2a(x - x_0)$$

5. Determine the surface area of a sphere in n-dimensional Euclidean space by carrying out the following steps:

- The surface area of an n-dimensional sphere of radius r is $S_n(r) = a_n r^{n-1}$, where a_n is independent of the radius r. Why?
- Show that

$$J = \int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

- Express J^n as an integral over Cartesian coordinates $(x_1, x_2, \cdots, x_n)$ in n-dimensional Euclidean space.
- Express J^n as an integral over the coordinate r, where $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$.
- Evaluate that integral over r, and thus determine the factor a_n in the surface area of an n-sphere.

Solution:
Let us begin by writing J^2 as a two-dimensional integral:

$$J^2 = \int_{-\infty}^{\infty} dx\, e^{-ax^2} \int_{-\infty}^{\infty} dy\, e^{-ay^2}$$

We transform this two-dimensional integral in polar coordinates:

$$J^2 = \int_0^{\infty} r\, e^{-ar^2}\, dr \int_0^{2\pi} d\theta = \frac{\pi}{a} \int_0^{\infty} d\left[-e^{-ar^2}\right] = \frac{\pi}{a}$$

Taking the positive square root, we determine $J = \sqrt{\pi/a}$. Next we express J^n as an integral over n-dimensional space, which we transform to polar coordinates, defining

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

We obtain

$$J^n = \int_0^\infty e^{-ar^2} \, S_n(r) \, dr$$

where $S_n(r)$ is the area of an n-dimensional spherical hypersurface of radius r. The volume of an n-dimensional sphere of radius R is given by this integral over Cartesian coordinates:

$$V_n(R) = \int_{x_1^2+x_2^2+\cdots+x_n^2 \le R^2} dx_1 dx_2 \cdots dx_n$$

Let us express this volume as an integral over scaled coordinates $t_i = x_i/R$:

$$V_n(R) = R^n \int_{t_1^2+t_2^2+\cdots+t_n^2 \le R^2} dt_1 dt_2 \cdots dt_n \equiv= v_n R^n$$

since the integration over t_i is independent of R. We may write this volume integral as

$$V_n(R) = \int_0^R dr S_n(r)$$

so that

$$S_n(r) = s_n r^{n-1} = n v_n r^{n-1}$$

Consequently,

$$J^n = \left[\frac{\pi}{a}\right]^{n/2} = s_n \int_0^\infty e^{-ar^2} \, r^{n-1} \, dr$$

We define the variable $u = ar^2$ to transform that integral:

$$J^n = \frac{s_n}{2a^{n/2}} \int u^{n/2-1} e^{-u} du$$

Using the definition of the Euler Gamma function,

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \, du$$

we obtain the result

$$J^n = \left[\frac{\pi}{a}\right]^{n/2} = \frac{s_n}{2a^{n/2}}\Gamma(n/2)$$

We may thus solve for s_n:

$$s_n = 2\,\frac{\pi^{n/2}}{\Gamma(n/2)}$$

The Euler Gamma function obeys the relation $\Gamma(z+1) = z\Gamma(z)$, as we show by integrating by parts:

$$\Gamma(z+1) = \int_0^\infty u^z e^{-u}\,du = \int_0^\infty u^z\,d[-e^{-u}] = \left|-u^z e^{-u}\right|_0^\infty + z\int_0^\infty u^{z-1}e^{-u}\,du = z\Gamma(z)$$

From the formulas $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$, we may calculate the Gamma function and s_n at any value of n. Here are the values up to $n = 12$.

n	s_n	numerical
1	2	2.000
2	2π	6.283
3	4π	12.566
4	$2\pi^2$	19.739
5	$\frac{8}{3}\pi^2$	26.319
6	π^3	31.006
7	$\frac{16}{15}\pi^3$	33.073
8	$\frac{1}{3}\pi^4$	32.470
9	$\frac{32}{105}\pi^4$	29.687
10	$\frac{1}{12}\pi^5$	25.502
11	$\frac{64}{245}\pi^5$	20.725
12	$\frac{1}{60}\pi^6$	16.023

The largest coefficient occurs at $n = 7$.

6. In a Schwarzschild metric, what is the volume contained in the spherical shell between Schwarzschild coordinates r_1 and r_2, where $r_2 > r_1 > R$, where R is the Schwarzschild radius?
Solution:
The Schwarzschild metric is

$$
\begin{aligned}
ds^2 &= ds_t^2 - ds_r^2 - ds_\theta^2 - ds_\phi^2 \\
&= (1-\frac{2R}{r})(cdt)^2 - \frac{dr^2}{1-2R/r} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)
\end{aligned}
$$

The volume of the region between radius $r = 2R$ and $r = r'$ is

$$
\begin{aligned}
V(r') &= \int ds_r \int ds_\theta \int ds_\phi = \int_{2R}^{r'} \frac{dr}{\sqrt{1-2R/r}} \int_0^\pi r\, d\theta \int_0^{2\pi} r\, \sin\theta\, d\phi \\
&= 4\pi \int_{2R}^{r'} \frac{r^2\, dr}{\sqrt{1-2R/r}}
\end{aligned}
$$

Let us define the variable $u = 2R/r$, so that $dr = -2Rdu/u^2$ and

$$
V(r') = 32\pi R^3 \int_{u'}^{1} \frac{du}{u^4} \frac{1}{\sqrt{1-u}}
$$

where the upper limit is $u' = 2R/r'$. We define the angular variable ρ by the relation $u = \cos^2\rho$, so that $du = -2\cos\rho\, \sin\rho\, d\rho$ and

$$
V = 64\pi R^3 \int_0^{\rho'} \sec^7\rho\, d\rho
$$

where $\cos^2\rho' = u' = 2R/r'$.
The integral may be evaluated in closed form [1]

$$
\int_0^{\rho'} \sec^7\rho\, d\rho = \tan\rho'\, \sec\rho' \left[\frac{1}{6}\sec^4\rho' + \frac{5}{24}\sec^2\rho' + \frac{5}{16}\right] + \frac{5}{16}\, \ln\left[\tan\rho' + \sec\rho'\right]
$$

Consequently, the volume $V(r')$ is

$$
V(r') = 4\pi \left[\sqrt{r'(r'-2R)}\left(\frac{1}{3}r'^2 + \frac{5}{6}r'R + \frac{5}{2}R^2\right) + \frac{5}{16}\, \ln\left[\left(\sqrt{\frac{r'}{2R}} + \sqrt{\frac{r'-2R}{2R}}\right)\right]\right]
$$

In the limit $r' >> 2R$, this volume approaches the "flat space" limit, $V(r') = 4\pi r'^3/3$. The volume between radii r_2 and r_1 is $V(r_2) - V(r_1)$.

[1] Gradshteyn and Ryzhik, Table of Integrals, Series, and Products (1965), 2.526.15, p. 137