1. Verify that the trajectory of a photon in the vicinity of a weak Schwarzschild metric has the following solution in the inverse radius $u = 1/r$:

$$u = \frac{\sin \phi}{b} + \frac{3GM}{2c^2 b^2} \left[ 1 + \frac{1}{3} \cos 2\theta \right]$$

to first order in the relativistic perturbation of a straight line path, with $GM \ll c^2 r$.

**Solution:**

The photon trajectory in the equatorial plane $\theta = \pi/2$ is a null geodesic, with the effective action

$$S = \int ds \left[ c^2 \left( \frac{dt}{ds} \right)^2 \left( 1 - \frac{2R}{r} \right) - \frac{(\frac{dr}{ds})^2}{1 - 2R/r} - r^2 \left( \frac{d\phi}{ds} \right)^2 \right]$$

The three constants of the motion are

$$r^2 \left( \frac{d\phi}{ds} \right) = B$$
$$\frac{dt}{ds} \left( 1 - \frac{2R}{r} \right) = A$$
$$c^2 \left( \frac{dt}{ds} \right)^2 \left( 1 - \frac{2R}{r} \right) - \frac{(\frac{dr}{ds})^2}{1 - 2R/r} - r^2 \left( \frac{d\phi}{ds} \right)^2 = 0$$

We substitute the first two equations into the third one to obtain

$$c^2 A^2 = \left( \frac{dr}{ds} \right)^2 + \frac{B^2}{r^2} \left( 1 - \frac{2R}{r} \right)$$

Then we use the relation

$$\frac{dr}{ds} = \frac{dr}{d\phi} \frac{d\phi}{ds}$$

and the first constant of the motion to obtain
\[
\frac{c^2 A^2}{B^2} = \frac{1}{r^4} \left( \frac{dr}{d\phi} \right)^2 + \frac{1}{r^2} \left( 1 - \frac{2R}{r} \right)
\]

We make the replacement \( u = 1/r \) to get

\[
\frac{c^2 A^2}{B^2} = \left( \frac{du}{d\phi} \right)^2 + u^2 \left( 1 - 2R u \right)
\]

Let us differentiate this relation with respect to \( \phi \), and cancel out the factor \( 2 \frac{du}{d\phi} \) to obtain

\[
\frac{d^2 u}{d\phi^2} + u = 3R_0 u^2
\]

When the right side of the differential equation is set to zero, the photon trajectory is a straight line, \( y = r \sin \phi = b \). We take this as an initial guess for the solution of the equation:

\[
u_0 = \frac{\sin \phi}{b}\]

Let us insert this in the right side of the differential equation, to get a refined guess:

\[
\frac{d^2 u_1}{d\phi^2} + u_1 = 3R_0 u_0^2 = \frac{3R_0}{b^2} \sin^2 \phi
\]

Using the method of variation of parameters, we write the solution as

\[
u_1 = f(\phi) \cos \phi + g(\phi) \sin \phi
\]

where

\[
\begin{align*}
f' \cos \phi + g' \sin \phi &= 0 \\
-f' \sin \phi + g' \cos \phi &= \frac{3R_0}{b^2} \sin^2 \phi
\end{align*}
\]

The solutions are
\[ f(\phi) = \frac{R_0}{b^2} (3 \cos \phi - \cos^3 \phi) + f_0 \]
\[ g(\phi) = \frac{R_0}{b^2} \sin^3 \phi + g_0 \]

Let us set the constants \( f_0 \) and \( g_0 \) to zero to obtain

\[ u_1(\phi) = \frac{R_0}{b^2} \left[ \sin^4 \phi - \cos^4 \phi + 3 \cos^2 \phi \right] \]
\[ = \frac{R_0}{b^2} \left[ 1 + \cos^2 \phi \right] \]
\[ = \frac{R_0}{2b^2} [3 + \cos 2\phi] \]

The result is established.

2. All massive objects look larger than they really are. Show that a light ray grazing the surface of a massive sphere of coordinate radius \( r > 3GM/c^2 \) will arrive at infinity with impact parameter \( b = r \sqrt{\frac{r}{r-2GM/c^2}} \).

Hence show that the apparent diameter of the Sun \([M = 2 \times 10^{30} \text{ kg} \text{ and } R = 7 \times 10^8 \text{ m}]\) exceeds its coordinate diameter by nearly 3 km.

**Solution:**

Let us begin with the equation for \( u(\phi) = 1/r(\phi) \) obtained in problem 1:

\[ \frac{c^2 A^2}{B^2} = \left( \frac{du}{d\phi} \right)^2 + u^2 (1 - 2R u) \]

At the distance of closest approach, \( r = r_0 \) we have \( du/d\phi = 0 \), so that

\[ \frac{c^2 A^2}{B^2} = u_0^2 (1 - 2R u_0) = \left[ 1 - \frac{2R}{r_0} \right] \frac{1}{r_0^2} \]

At great distance from the sphere, we have \( u = 0 \) and \( du/d\phi = 1/b \), so that

\[ \frac{c^2 A^2}{B^2} = \frac{1}{b^2} \]
Thus,

\[
\frac{1}{b^2} = \frac{1}{r_0} \frac{1 - \frac{2R}{r_0}}{r_0} \\
\frac{b^2}{r_0} = \frac{r_0^2}{1 - 2R/r_0} \\
b = \frac{r_0}{\sqrt{1 - 2R/r_0}}
\]

3. The nearest star appears to have a brightness (energy flux) of \(10^{-11}\) of the sun. Assuming that it has the same luminosity as our sun, determine the distance to the star.

Note: 1 AU [astronomical unit] is about \(5 \times 10^{-6}\) parsecs.

Solution:
The brightness \(L\) of an isotropic star is the power radiated \((P)\) per unit area \((4\pi R^2)\).

\[P = L_0 \cdot (4\pi R_0^2) = L \cdot (4\pi R^2)\]

Thus

\[R^2 = R_0^2 \cdot \frac{L_0}{L} = \frac{R_0^2}{10^{-11}} = 10^{11} R_0^2\]

\[R = 3.2 \times 10^5 \cdot R_0 = 3.2 \times 10^5 \cdot 5 \times 10^{-6} \text{ parsec} = 1.6 \text{ parsec}\]

4. A beam of photons with circular cross section of radius \(a\) is aimed toward a black hole of mass \(M\) from far away. The center of the beam is aimed at the center of the hole. What is the largest radius \(a = a_{\text{max}}\) such that all the photons in the beam are captured by the black hole? The capture cross section is \(\pi a_{\text{max}}^2\).

Solution:
As in Problem 1, the photon path is a null geodesic with the effective action

\[S = \int ds \left[ c^2 \left( \frac{dt}{ds} \right)^2 - \frac{\left( \frac{dr}{ds} \right)^2}{1 - 2R/r} - r^2 \left( \frac{d\phi}{ds} \right)^2 \right]\]

We use the three constants of motion given in Problem 1 to obtain
\[ c^2 A^2 = \left( \frac{dr}{ds} \right)^2 + B^2 \frac{r^2}{r^2} \left( 1 - \frac{2R}{r} \right) \]

The photon comes in from infinity with impact parameter \( b \), so that at large \( r \)

\[
\begin{align*}
    b &= r \sin \phi \approx r \phi \\
    \dot{\phi} &\approx -\frac{br}{r^2} \approx \frac{cb}{r^2} \\
    r^2 \dot{\phi} &= \frac{B}{A} = cb
\end{align*}
\]

Thus

\[
\frac{1}{b^2} = \frac{1}{B^2} \left( \frac{dr}{ds} \right)^2 + \frac{1}{r^2} \left( 1 - \frac{2R}{r} \right)
\]

This corresponds to motion in an effective potential \( V_{\text{eff}}(r) \):

\[
\begin{align*}
    \frac{1}{b^2} &= \frac{1}{B^2} \left( \frac{dr}{ds} \right)^2 + V_{\text{eff}}(r) \\
    V_{\text{eff}}(r) &= \frac{1}{r^2} - \frac{2R}{r^3}
\end{align*}
\]

This effective potential has its minimum value at \( r = 3R \):

\[
\frac{dV_{\text{eff}}}{dr} = \frac{-2}{r^3} + \frac{6R}{r^4} = \begin{cases} 0 \quad & r - 3R = 0 \end{cases}
\]

The minimum value of the effective potential is

\[
V_{\text{eff}}(3R) = \frac{1}{9R^2} - \frac{2R}{27R^3} = \frac{1}{27R^2}
\]

For the case in which

\[
V_{\text{eff}}(3R) = \frac{1}{27R^2} < \frac{1}{b^2}
\]
the incident photon is absorbed by the black hole. That is, for impact parameters \( b < \sqrt{27R} \), the photon is absorbed.

5. A particle is to be launched in the outward radial direction from the point \( r = 4GM/c^2 \) in the Schwarzschild geometry.

- At what speed \( dr/dt \) must the particle be launched if it is to reach the point \( r = 8GM/c^2 \) with zero speed?

- How much proper time does this trip take?

Solution:
The geodesic trajectory is determined from the principal of least action, with the effective action chosen as

\[
S = \int ds \left[ c^2 \left( \frac{dt}{ds} \right)^2 \left( 1 - \frac{2R}{r} \right) - \frac{(dr/ds)^2}{1 - 2R/r} \right]
\]

We obtain two constants of the motion, because the parameter \( t \) and the path length \( s \) do not appear in the effective Lagrangian:

\[
\frac{dt}{ds} \left( 1 - \frac{2R}{r} \right) = A
\]

\[
c^2 \left( \frac{dt}{ds} \right)^2 \left( 1 - \frac{2R}{r} \right) - \frac{(dr/ds)^2}{1 - 2R/r} = 1
\]

We use the relation

\[
\frac{dr}{ds} = \frac{dr}{dt} \frac{dt}{ds}
\]

and substitute the first constant into the second to obtain

\[
A^2 \left[ c^2 - \frac{(dr/dt)^2}{(1 - 2R/r)^2} \right] = 1 \frac{2R}{r}
\]

We set \( dr/dt = 0 \) at \( r = 8R \) to determine the constant \( A \):

\[
A^2 c^2 = \frac{3}{4}
\]
Thus we obtain

\[
\left( \frac{dr}{dt} \right)^2 = \frac{c^2}{3} \left( 1 - \frac{2R}{r} \right)^2 \left( \frac{8R}{r} - 1 \right)
\]

The velocity at \( r = 4R \) is given by \( \frac{dr}{dt} = c/\sqrt{12} = 0.29 \, c \). The proper time \( d\tau \) to travel a distance \( dr \) is given by

\[
(c d\tau)^2 = c^2 \left( 1 - \frac{2R}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2R}{r}}
\]

Thus we obtain

\[
(c d\tau)^2 = \frac{dr^2}{1 - \frac{2R}{r}} \left[ \frac{c^2 \left( 1 - \frac{2R}{r} \right)^2}{(dr/dt)^2} - 1 \right]
\
= \frac{dr^2}{1 - \frac{2R}{r}} \left[ \frac{3}{\frac{8R}{r} - 1} - 1 \right]
\
= \frac{4 \, dr^2}{\frac{8R}{r} - 1}
\]

The proper time to travel from \( r = 8R \) to \( r = 4R \) is thus given by

\[
c\tau = 2 \int_{4R}^{8R} \frac{dr}{\sqrt{\frac{8R}{r} - 1}}
\]

Let us change the variable of integration to \( u \), with \( u^2 = \frac{8R}{r} - 1 \):

\[
c\tau = 32 \, R \int_{0}^{1} \frac{du}{(1 + u^2)^2}
\]

This integral is evaluated by letting \( u = \tan \theta \):

\[
c\tau = 32 \, R \int_{0}^{\pi/4} d\theta \cos^2 \theta = 4 \, (\pi + 2) \, R
\]