Physics 403: Relativity

Homework Assignment 6

Due 23 April 2007

1. Verify that the trajectory of a photon in the vicinity of a weak Schwarzschild metric has the following solution in the inverse radius $u = 1/r$:

$$u = \frac{\sin\phi}{b} + \frac{3GM}{2c^2b^2}\left[1 + \frac{1}{3}\cos 2\theta\right]$$

to first order in the relativistic perturbation of a straight line path, with $GM \ll c^2 r$.
<u>Solution</u>:
The photon trajectory in the equatorial plane $\theta = \pi/2$ is a null geodesic, with the effective action

$$S = \int ds \left[c^2\left(\frac{dt}{ds}\right)^2\left(1 - \frac{2R}{r}\right) - \frac{\left(\frac{dr}{ds}\right)^2}{1 - 2R/r} - r^2\left(\frac{d\phi}{ds}\right)^2\right]$$

The three constants of the motion are

$$\begin{aligned} r^2\left(\frac{d\phi}{ds}\right) &= B \\ \frac{dt}{ds}\left(1 - \frac{2R}{r}\right) &= A \\ c^2\left(\frac{dt}{ds}\right)^2\left(1 - \frac{2R}{r}\right) - \frac{\left(\frac{dr}{ds}\right)^2}{1 - 2R/r} - r^2\left(\frac{d\phi}{ds}\right)^2 &= 0 \end{aligned}$$

We substitute the first two equations into the third one to obtain

$$c^2A^2 = \left(\frac{dr}{ds}\right)^2 + \frac{B^2}{r^2}\left(1 - \frac{2R}{r}\right)$$

Then we use the relation

$$\frac{dr}{ds} = \frac{dr}{d\phi}\frac{d\phi}{ds}$$

and the first constant of the motion to obtain

$$\frac{c^2A^2}{B^2} = \frac{1}{r^4}\,\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2}\,\left(1 - \frac{2R}{r}\right)$$

We make the replacement $u = 1/r$ to get

$$\frac{c^2A^2}{B^2} = \left(\frac{du}{d\phi}\right)^2 + u^2\ (1 - 2R\,u)$$

Let us differentiate this relation with respect to ϕ, and cancel out the factor $2\,du/d\phi$ to obtain

$$\frac{d^2u}{d\phi^2} + u = 3\,R_0\,u^2$$

When the right side of the differential equation is set to zero, the photon trajectory is a straight line, $y = r\sin\phi = b$. We take this as an initial guess for the solution of the equation:

$$u_0 = \frac{\sin\phi}{b}$$

Let us insert this in the right side of the differential equation, to get a refined guess:

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3\,R_0\,u_0^2 = \frac{3R_0}{b^2}\sin^2\phi$$

Using the method of variation of parameters, we write the solution as

$$u_1 = f(\phi)\ \cos\phi + g(\phi)\ \sin\phi$$

where

$$\begin{aligned} f'\cos\phi + g'\sin\phi &= 0 \\ -f'\sin\phi + g'\cos\phi &= \frac{3R_0}{b^2}\sin^2\phi \end{aligned}$$

The solutions are

$$\begin{aligned} f(\phi) &= \frac{R_0}{b^2}\left(3\cos\phi - \cos^3\phi\right) + f_0 \\ g(\phi) &= \frac{R_0}{b^2}\sin^3\phi + g_0 \end{aligned}$$

Let us set the constants f_0 and g_0 to zero to obtain

$$\begin{aligned} u_1(\phi) &= \frac{R_0}{b^2}\left[\sin^4\phi - \cos^4\phi + 3\cos^2\phi\right] \\ &= \frac{R_0}{b^2}\left[1 + \cos^2\phi\right] \\ &= \frac{R_0}{2b^2}\left[3 + \cos 2\phi\right] \end{aligned}$$

The result is established.

2. All massive objects look larger than they really are. Show tht a light ray grazing the surface of a massive sphere of coordinate radius $r > 3GM/c^2$ will arrive at infinity with impact parameter

$$b = r\sqrt{\frac{r}{r - 2GM/c^2}}$$

Hence show that the apparent diameter of the Sun [$M = 2 \times 10^{30}$ kg and $R = 7 \times 10^8$ m] exceeds its coordinate diameter by nearly 3 km.

Solution:

Let us begin with the equation for $u(\phi) = 1/r(\phi)$ obtained in problem 1:

$$\frac{c^2A^2}{B^2} = \left(\frac{du}{d\phi}\right)^2 + u^2\ (1 - 2R\,u)$$

At the distance of closest approach, $r = r_0$ we have $du/d\phi = 0$, so that

$$\frac{c^2A^2}{B^2} = u_0^2(1 - 2R\,u_0) = \left[1 - \frac{2R}{r_0}\right]\frac{1}{r_0^2}$$

At great distance from the sphere, we have $u = 0$ and $du/d\phi = 1/b$, so that

$$\frac{c^2A^2}{B^2} = \frac{1}{b^2}$$

Thus,

$$\begin{aligned}
\frac{1}{b^2} &= \left[1-\frac{2R}{r_0}\right]\frac{1}{r_0^2} \\
b^2 &= \frac{r_0^2}{1-2R/r_0} \\
b &= \frac{r_0}{\sqrt{1-2R/r_0}}
\end{aligned}$$

3. The nearest star appears to have a brightness (energy flux) of 10^{-11} of the sun. Assuing that it has the same luminosity as our sun, determine the distance to the star.
Note: 1 AU [astronomical unit] is about 5×10^{-6} parsecs.
Solution:
The brightness L of an isotropic star is the power radiated (P) per unit area ($4\pi R^2$).

$$P = L_0\,(4\pi R_0^2) = L\,(4\pi R^2)$$

Thus

$$\begin{aligned}
R^2 &= R_0^2\,\frac{L_0}{L} = \frac{R_0^2}{10^{-11}} = 10^{11}\,R_0^2 \\
R &= 3.2\times10^5\cdot R_0 = 3.2\times10^5\cdot5\times10^{-6}\;parsec = 1.6\;parsec
\end{aligned}$$

4. A beam of photons with circular cross section of radius a is aimed toward a black hole of mass M from far away. The center of the beam is aimed at the center of the hole. What is the largest radius $a = a_{max}$ such that all the photons in the beam are captured by the black hole? The capture cross section is πa_{max}^2,
Solution:
As in Problem 1, the photon path is a null geodesic with the effective action

$$S = \int ds\left[c^2\left(\frac{dt}{ds}\right)^2\left(1-\frac{2R}{r}\right)-\frac{\left(\frac{dr}{ds}\right)^2}{1-2R/r}-r^2\left(\frac{d\phi}{ds}\right)^2\right]$$

We use the three constants of motion given in Problem 1 to obtain

$$c^2A^2 = \left(\frac{dr}{ds}\right)^2 + \frac{B^2}{r^2}\left(1 - \frac{2R}{r}\right)$$

The photon comes in from infinity with impact parameter b, so that at large r

$$\begin{aligned} b &= r\sin\phi \approx r\phi \\ \dot{\phi} &\approx -\frac{b\dot{r}}{r^2} \approx \frac{cb}{r^2} \\ r^2\dot{\phi} &= \frac{B}{A} = cb \end{aligned}$$

Thus

$$\frac{1}{b^2} = \frac{1}{B^2}\left(\frac{dr}{ds}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2R}{r}\right)$$

This corresponds to motion in an effective potential $V_{eff}(r)$:

$$\begin{aligned} \frac{1}{b^2} &= \frac{1}{B^2}\left(\frac{dr}{ds}\right)^2 + V_{eff}(r) \\ V_{eff}(r) &= \frac{1}{r^2} - \frac{2R}{r^3} \end{aligned}$$

This effective potential has its minimum value at $r = 3R$:

$$\begin{aligned} \frac{dV_{eff}}{dr} = \frac{-2}{r^3} + \frac{6R}{r^4} &= 0 \\ r - 3R &= 0 \end{aligned}$$

The minimum value of the effective potential is

$$V_{eff}(3R) = \frac{1}{9R^2} - \frac{2R}{27R^3} = \frac{1}{27R^2}$$

For the case in which

$$V_{eff}(3R) = \frac{1}{27R^2} < \frac{1}{b^2}$$

the incident photon is absorbed by the black hole. That is, for impact parameters $b < \sqrt{27}R$, the photon is absorbed.

5. A particle is to be launched in the outward radial direction from the point $r = 4GM/c^2$ in the Schwarzschild geometry.

- At what speed dr/dt must the particle be launched if it is to reach the point $r = 8GM/c^2$ with zero speed?
- How much proper time does this trip take?

Solution:
The geodesic trajectory is determined from the principal of least action, with the effective action chosen as

$$S = \int ds \left[c^2 \left(\frac{dt}{ds}\right)^2 \left(1 - \frac{2R}{r}\right) - \frac{\left(\frac{dr}{ds}\right)^2}{1 - 2R/r} \right]$$

We obtain two constants of the motion, because the parameter t and the path length s do not appear in the effective Lagrangian:

$$\begin{aligned} \frac{dt}{ds}\left(1 - \frac{2R}{r}\right) &= A \\ c^2 \left(\frac{dt}{ds}\right)^2 \left(1 - \frac{2R}{r}\right) - \frac{\left(\frac{dr}{ds}\right)^2}{1 - 2R/r} &= 1 \end{aligned}$$

We use the relation

$$\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds}$$

and substitute the first constant into the second to obtain

$$A^2 \left[c^2 - \frac{\left(\frac{dr}{dt}\right)^2}{\left(1 - \frac{2R}{r}\right)^2} \right] = 1 - \frac{2R}{r}$$

We set $dr/dt = 0$ at $r = 8R$ to determine the constant A:

$$A^2\, c^2 = \frac{3}{4}$$

Thus we obtain

$$\left(\frac{dr}{dt}\right)^2 = \frac{c^2}{3}\left(1-\frac{2R}{r}\right)^2\left(\frac{8R}{r}-1\right)$$

The velocity at $r = 4\,R$ is given by $dr/dt = c/\sqrt{12} = 0.29\,c$. The proper time $d\tau$ to travel a distance dr is given by

$$(cd\tau)^2 = c^2\left(1-\frac{2R}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2R}{r}}$$

Thus we obtain

$$\begin{aligned}(cd\tau)^2 &= \frac{dr^2}{1-\frac{2R}{r}}\left[\frac{c^2\left(1-\frac{2R}{r}\right)^2}{\left(\frac{dr}{dt}\right)^2}-1\right]\\ &= \frac{dr^2}{1-\frac{2R}{r}}\left[\frac{3}{\frac{8R}{r}-1}-1\right]\\ &= \frac{4\,dr^2}{\frac{8R}{r}-1}\end{aligned}$$

The proper time to travel from $r = 8\,R$ to $r = 4\,R$ is thus given by

$$c\tau = 2\int_{4R}^{8R}\frac{dr}{\sqrt{\frac{8R}{r}-1}}$$

Let us change the variable of integration to u, with $u^2 = 8R/r - 1$:

$$c\tau = 32\,R\int_0^1\frac{du}{(1+u^2)^2}$$

This integral is evaluated by letting $u = \tan\theta$:

$$c\tau = 32\,R\int_0^{\pi/4} d\theta\,\cos^2\theta = 4\,(\pi+2)\,R$$