Radial Coulomb field from a charge $q$ at rest, at distance $\vec{r}$:

$$\vec{E}_r = \frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2}$$

We assume that the electric field is continuously transmitted from the charge into space with velocity $c$. If the charge is undergoing an instantaneous acceleration to some small velocity $\Delta v$ over a short time $\Delta t$, the electric field develops a small tangential component at distance $r$ after a time $t = r/c$.

$$\frac{E_t}{E_r} = \frac{\Delta v t \sin \theta}{c \Delta t}$$

where $\theta$ is the angle between the acceleration vector and the radial direction. Equivalently

$$E_t = E_r \frac{\Delta v}{\Delta t} \frac{r \sin \theta}{c^2} = \frac{q}{4\pi\varepsilon_0} \frac{\sin \theta}{rc^2}$$

Note that the transverse electric field varies inversely with the distance to the point in question. It follows from Faraday’s law that there is a corresponding magnetic field of magnitude

$$B_t = E_t / c$$

which is perpendicular to the radial direction and perpendicular to the tangential electric field. This geometrical construction of the radiation field is credited to J. J. Thomson, who also discovered the electron and the proton.

Let the charge experience an acceleration $\vec{a}$ in the $z$-direction. The fields at an angle $\theta$ to the $z$-direction are given by

$$E_0 = \frac{qa}{4\pi\varepsilon_0 c^2} \frac{\sin \theta}{r}$$

$$B_0 = E_0 / c$$

The electric field $\vec{E}$ lies in the plane formed by $\vec{r}$ and $\vec{a}$, and perpendicular to $\vec{r}$, whereas the magnetic field $\vec{B}$ is perpendicular to that plane.
The Poynting Vector is given by

\[ \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{E_0^2}{\mu_0 c} \hat{r} = \frac{q^2 a^2}{(4\pi \varepsilon_0 c^2)^2 \mu_0 c r^2} \hat{r} \]

The total energy radiated per unit time through a sphere of radius \( r \) is

\[ P = \frac{dE}{dt} = \oint \vec{S} \cdot d\vec{A} = r^2 \oint S d\Omega = \frac{q^2 a^2 \varepsilon_0}{(4\pi \varepsilon_0 c^2) c^3} \oint \sin^2 \theta d\Omega \]

The integral in the formula is

\[ \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\phi = \frac{4}{3} \cdot 2\pi = \frac{8\pi}{3} \]

Thus we obtain

\[ P = \frac{2}{3} \frac{q^2 a^2}{4\pi \varepsilon_0 c^3} \]

For an electron (charge \( e_0 = -1.6 \times 10^{-19} \text{ C} \) placed in an electric field of magnitude 1 Volt per meter, the acceleration is \( a = e_0 E/m = 1.8 \times 10^{11} \text{ m/s}^2 \). The power radiated in electromagnetic energy in this case is \( P = 2 \times 10^{-31} \text{ Watts} \), for each electron.

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\(^1\text{Recall that } \mu_0 \varepsilon_0 c^2 = 1.\)