Electromagnetic Radiation from Accelerated Charges
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Fields at distance $\vec{r}$ for charge q moving at constant velocity $\vec{v}$:

$$\begin{aligned}\vec{E} &= \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\\ \vec{B} &= \frac{\mu_0 q}{4\pi}\frac{\vec{v}\times\hat{r}}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{v}\times\hat{r}}{c^2r^2}\end{aligned}$$

In the last step we have used the relation $\varepsilon_0\mu_0 = 1/c^2$.
Let the charge experience an acceleration $\vec{a}$ in the z-direction. The fields at an angle θ to the z-direction are given by

$$\begin{aligned}E_0 &= \frac{qa}{4\pi\varepsilon_0 c^2}\frac{\sin\theta}{r}\\ B_0 &= E_0/c\end{aligned}$$

The electric field $\vec{E}$ lies in the plane formed by $\vec{r}$ and $\vec{a}$, and perpendicular to $\vec{r}$, whereas the magnetic field $\vec{B}$ is perpendicular to that plane.
The Poynting Vector is given by

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{E_0^2}{\mu_0 c}\,\hat{r} = \frac{q^2a^2}{(4\pi\varepsilon_0 c^2)^2}\frac{\sin^2\theta}{\mu_0 c r^2}\,\hat{r}$$

The total energy radiated per unit time through a sphere of radius r is

$$P = \frac{d\mathcal{E}}{dt} = \oint \vec{S}\cdot\vec{dA} = r^2\oint S\,d\Omega = \frac{q^2a^2\varepsilon_0}{(4\pi\varepsilon_0)^2c^3}\oint \sin^2\theta\, d\Omega$$

The integral in the formula is

$$\int_0^{\pi} d\theta \sin^3\theta \int_0^{2\pi} d\phi = \frac{4}{3}\cdot 2\pi = \frac{8\pi}{3}$$

Thus we obtain

$$P = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0}\frac{a^2}{c^3}$$

For an electron (charge $e_0 = -1.6\times 10^{-19}$ C placed in an electric field of magnitude 1 Volt per meter, the acceleration is $a = e_0E/m = 1.8\times 10^{11}\ m/s^2$. The power radiated in electromagnetic energy in this case is $P = 2\times 10^{-31}$ Watts, for each electron.