More Curved Spirolaterals

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Abstract

Spirolaterals are square spirals constructed of straight lines of increasing length. This paper investigates methods that can be used to transform their straight line turns into curves. From the tens of thousands of possible spirolaterals which have been found, that are all closed, with and without reversed turns, a series of artworks is developed showing possible curving methods: curve fitting, splines, inversion, antiMercator and circular transformations, along with hypocycloid and epicycloid curves. These transformations are implemented using CAD methods and custom software. A series of parameters are discussed which generate a variety of artwork.

1. Introduction

A spirolateral is created by drawing a set of lines; the first at a unit length, then each additional line is increased by one unit length while turning to a constant angle. To complete a closed spirolateral, it is necessary to only repeat this procedure until the starting point is reached. The first apparent reference to this geometric figure was by Odds [8]. Further information can be found at Abelson [1] and Gardner [3]. In addition to the property of closure, spirolaterals need not always turn the same direction. The direction can be reversed at any turn, which makes the total number of possible spirolateral unknown.


2_{45} $\qquad$ 4_{60} $\qquad$ 4_{120}
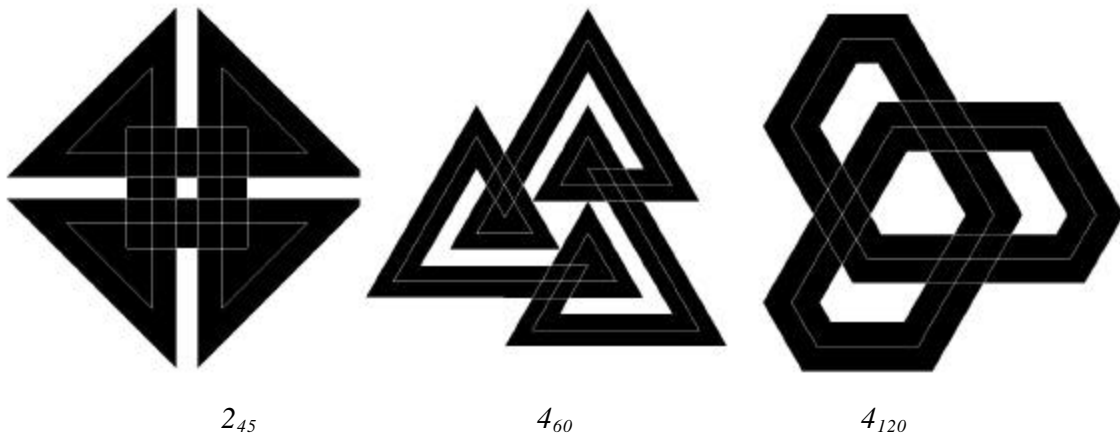
Figure 1: *Closed spirolaterals*

In this series of investigations, spirolaterals were generated to determine predictable closure and the variety of angles that could be used for turns as in Krawczyk [4]. The image of the spirolaterals remained as single lines. Reversals were the next area of interest. Krawczyk [6] covers the generation of all the unique reversals of any particular spirolateral. While developing software for the generation of spirolaterals as artwork, the single line was re-evaluated as the only representation. The turning line was given a thickness, and options were included to overlay a center line, edge lines, or corner lines. These

options were included in Krawczyk [5] and the web site, www.netcom.com/~bitart, which displays 200 spirolaterals. The web site also includes a JAVA program to generate your own variations. Figure 1 displays a sample of closed spirolaterals, and Figure 2 ones with reversed turns.

The notation for identifying spirolaterals starts with the number of turns, followed by the turning angle, then a list of the reversed turns, if present.

$$5_{90}^{1,3,4,6,7} \qquad 4_{60}^{1,3,4} \qquad 3_{36}^{1,3}$$
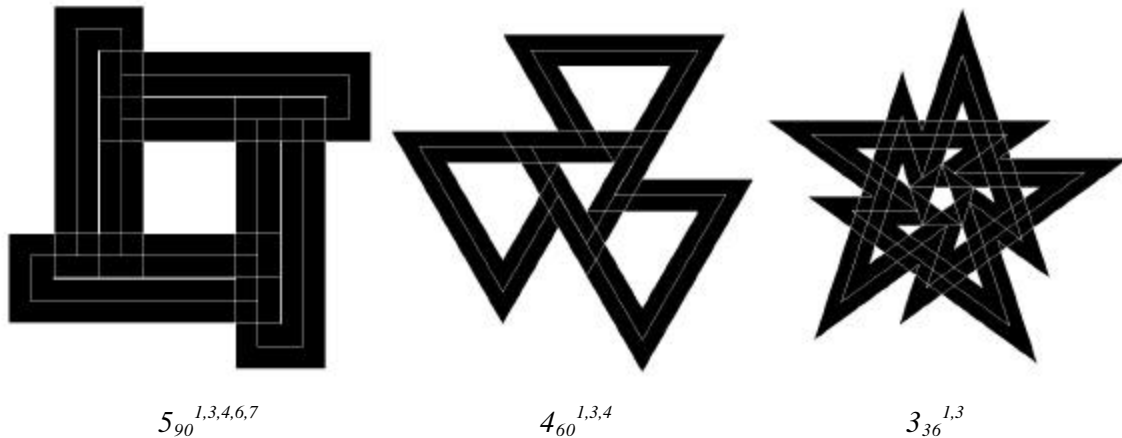
Figure 2: *Closed spirolaterals with reversals*

2. Curved Spirolaterals

One of the extensions of using spirolaterals as artwork beyond their original image was first discussed in Krawczyk [5]. A spirolateral consists of lines drawn between two vertices. Instead of using the actual lines, one idea was to use only the vertices. Using CAD-based software, procedures were written to interpret the vertices as control points for curves. Figure 3 displays this concept. The simple 1_{90} spirolateral, a square, consists of four vertices. These vertices can be fitted with arcs at the start and end of adjacent vertices. The vertices can also be control points to generate a spline curve.


Original   *Vertices*   *Curve Fit*   *Spline*
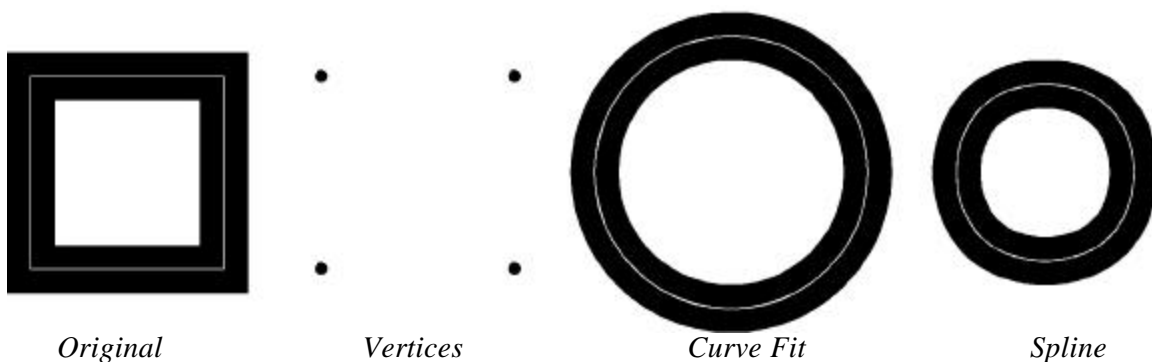
Figure 3: *Curved spirolateral 1_{90}*

Many of the spirolaterals developed as curves were found to be interesting but not visually exciting nor did they generate any unexpected images. The first set created were ones that consisted on ten or less turns. Figure 4 displays one of these.
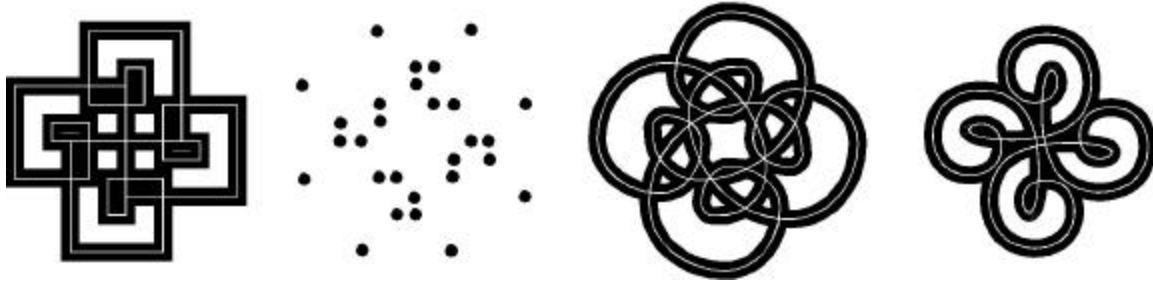
Figure 4: *Curved spirolateral 7_{90}*

Only when the number of turns increased to a range of 11 to 15, with reversed turns included, did the number of control points increase to the point where they begin to generate images of unexpected detail and complexity. Figure 5 displays two such examples.
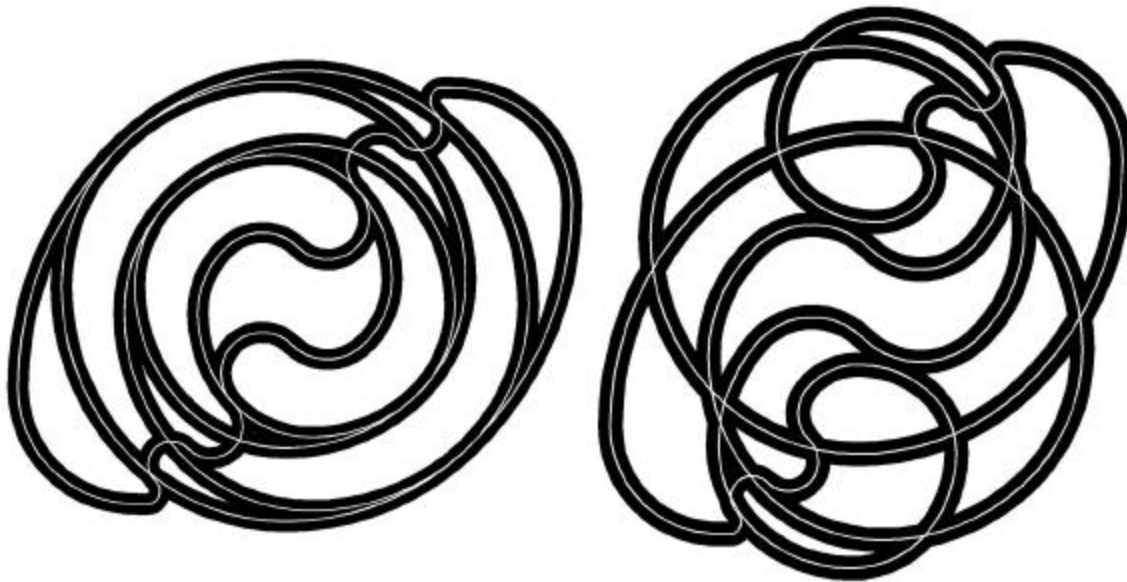
Figure 5: *Curved spirolaterals of 14_{90}*

3. Curves by Inversion

Another approach to curving spirolaterals is to apply a transformation, a projection, or mapping from a linear image to a curved one. In researching these possibilities, the first one encountered that was interesting was the concept of inversion, as described by Weisstein [9], Dixon [2] and Lawerence [7]. This mathematical method turns lines into circles. The transformation based on the image midpoint is as follows:
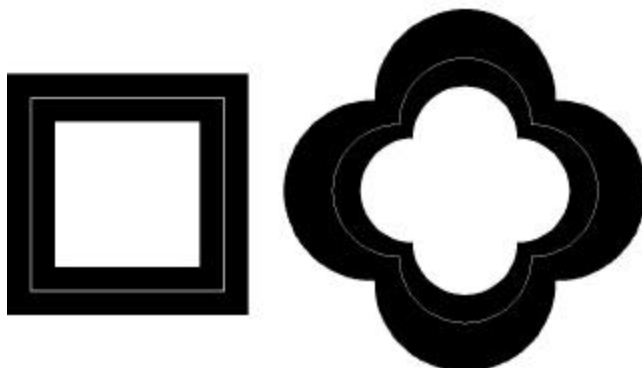
$$X = (x * r^2) / (x^2 + y^2)$$
$$Y = (y * r^2) / (x^2 + y^2)$$

Figure 6 demonstrates this transformation on a simple square spirolateral.

Inversion by definition reconstructs the spirolateral as curves. After transforming a variety of spirolaterals, it was found that ones that are open in the center generate the most interesting results. Ones that have lines that cross the image midpoint generate very strange results, which are visually not

interesting. Symmetry also added to the quality of the results, as did the selection of the line thickness. Figure 7 displays a sample of spirolateral inversions.

Figure 6: *Inversion of spirolateral 1_{90}*

$7_{90}^{1,3,4,6,7}$ 2_{45} 2_{30}
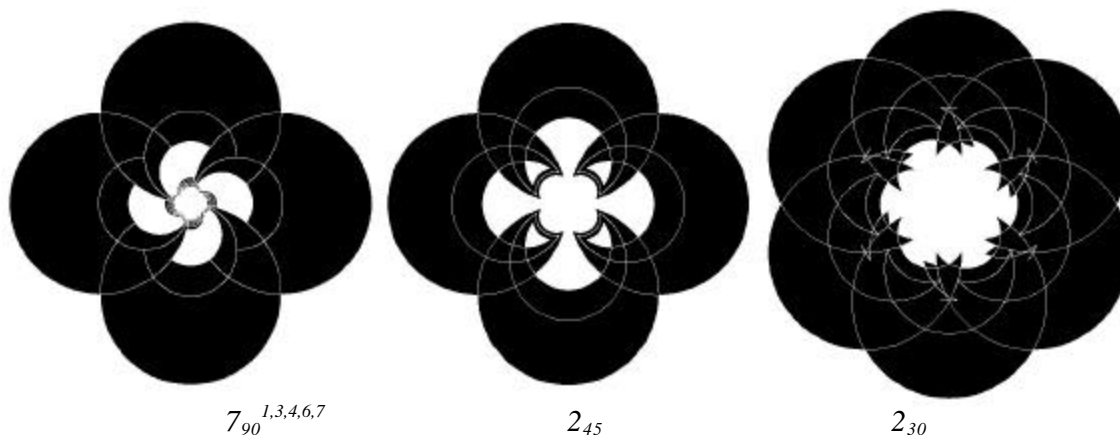
Figure 7: *Spirolateral inversions*

4.   Circular Transformations

Continuing the search for other methods of curving spirolaterals, Dixon [2] in a section on transformations outlines a method he calls anitMercator. Horizontal lines become circles concentric with the coordinate origin, vertical lines become radial, and slanting lines become logarithmic spirals. The transformation in polar coordinates is as follows:

A = k * x   where: k = 2π/(xmax-xmin)
R = exp(k * y)

Figure 8 demonstrates this transformation on a simple square spirolateral.

This transformation is affected by the line thickness and also by the offset from the origin. The origin is positioned in the lower-left corner of the image. This location results in the image being bent clockwise starting with the left corner.

Figure 9 displays a sample of spirolateral transformed by antiMercator.
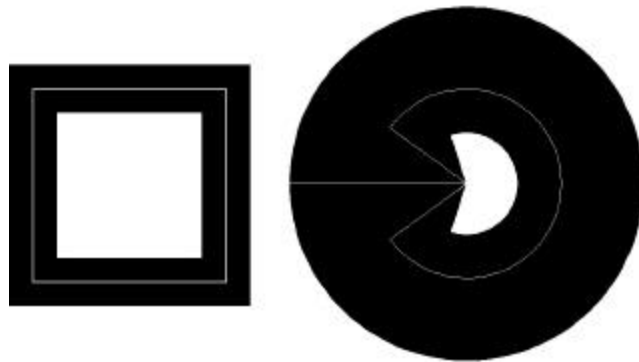
Figure 8: *antiMercator spirolateral 1_{90}*

1_{60} 2_{30} 3_{90}
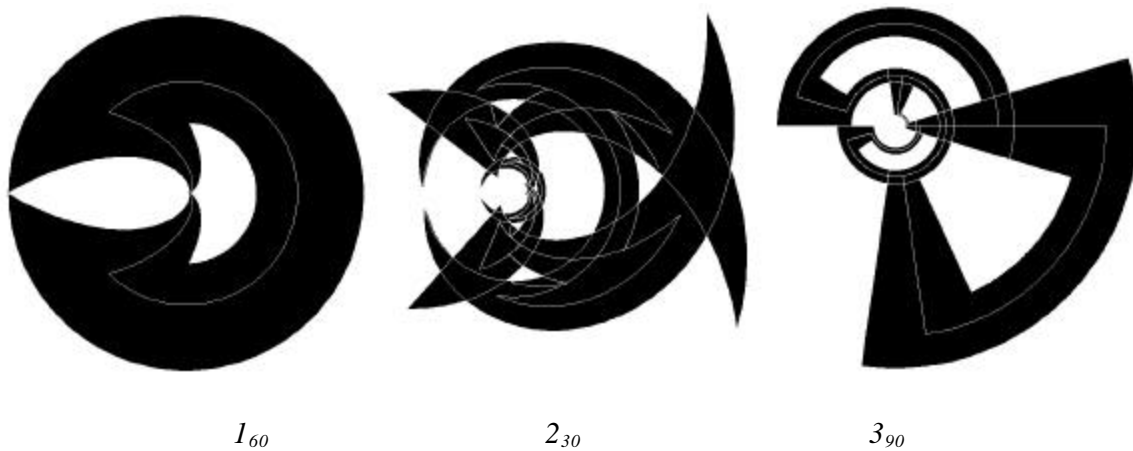
Figure 9: *antiMercator spirolaterals*

While investigating the antiMercator transformation, one alternate method was found by removing the exponential function. The circular form remains without the logarithmic spiral effect. Since no formal name has been found for this transformation, it will be referred to as simply Circular. Figure 10 demonstrates this circular transformation on a simple square spirolateral.

This transformation differs from the antiMercator in that the line spacing is of a more constant distance from the center, so that the original distances are better represented. This transformation also changes as the offset increases. Figure 11 displays a sample of spirolateral transformed by the Circular transformation.
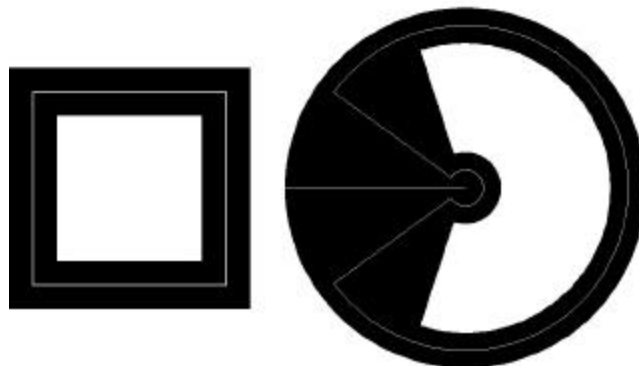
Figure 10: *Circular spirolateral 1_{90}*

1_{60} $\qquad$ 2_{45} $\qquad$ 2_{30}
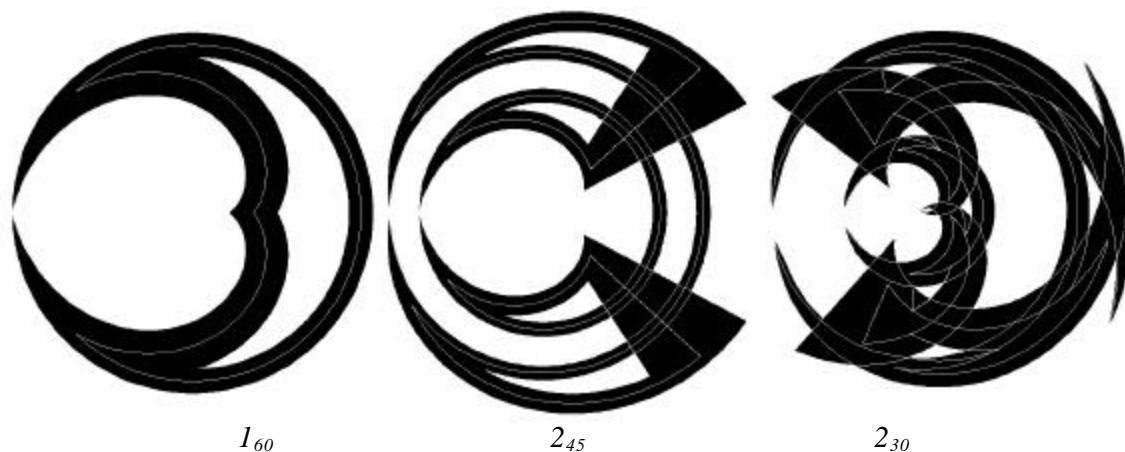
Figure 11: *Circular spirolaterals*

5. Following a Curve

The antiMercator and Circular transformation consist of computing polar coordinates based on the original image and then converting them to Cartesian coordinates. An alternate method is to not convert the polar to rectangular Cartesian coordinates but to follow curved ones. Using the angle and radius computation from the Circular transformation, the modified Cartesian coordinates are computed with the following curves, as in Lawerence [7]:

Hypocycloid: $X = R * (\cos(A)*(n-1) + \cos(A*(n-1)))$ where: n = number of cusps, 3 and 4
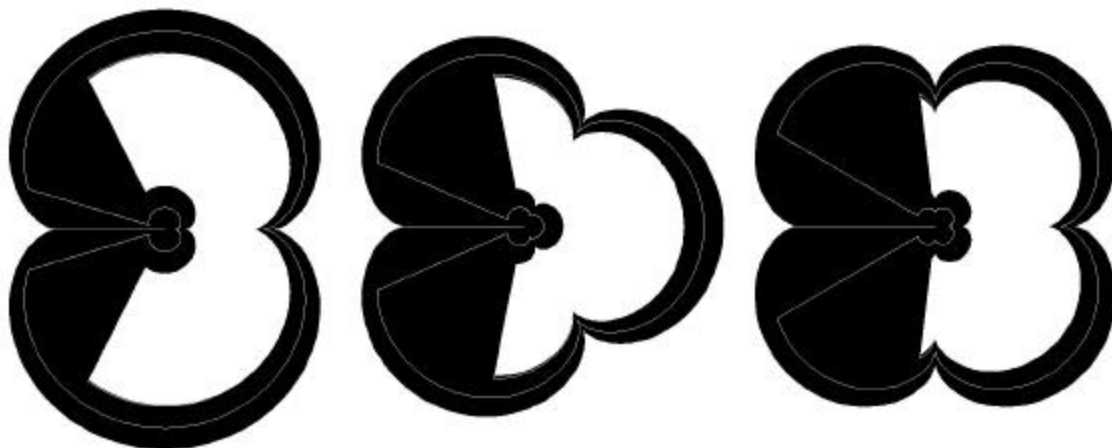$\qquad\qquad\quad$ $Y = R * (\sin(A)*(n-1) - \sin(A*(n-1)))$

Epicycloid: $X = R * (\cos(A)*(n+1) - \cos(A*(n+1)))$ where: n = number of cusps, 2, 3, and 4
$\qquad\qquad\quad$ $Y = R * (\sin(A) (n+1) - \sin(A*(n+1)))$

The major difference between these two curves is that the Hypocycloid produces concave curves and the Epicycloid produces convex ones.

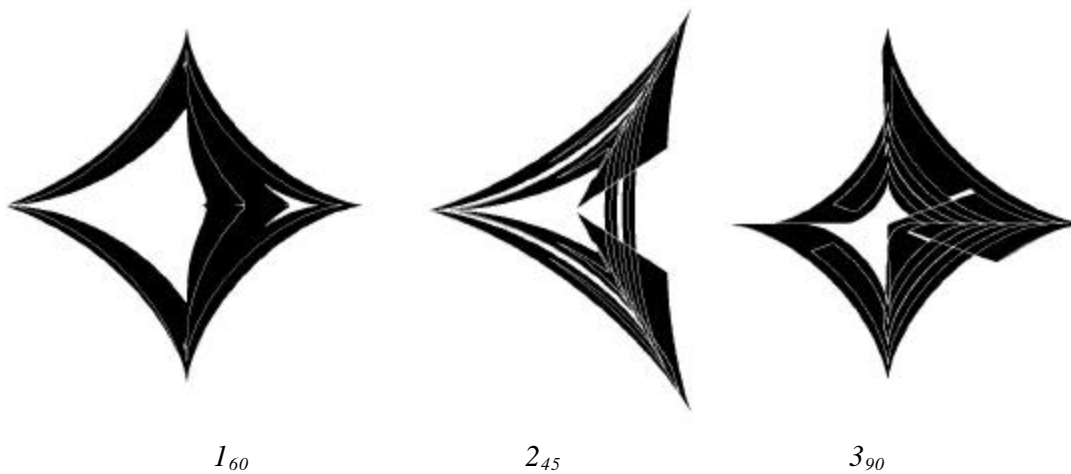Figures 12 and 13 demonstrate these transformations on a simple square spirolateral.
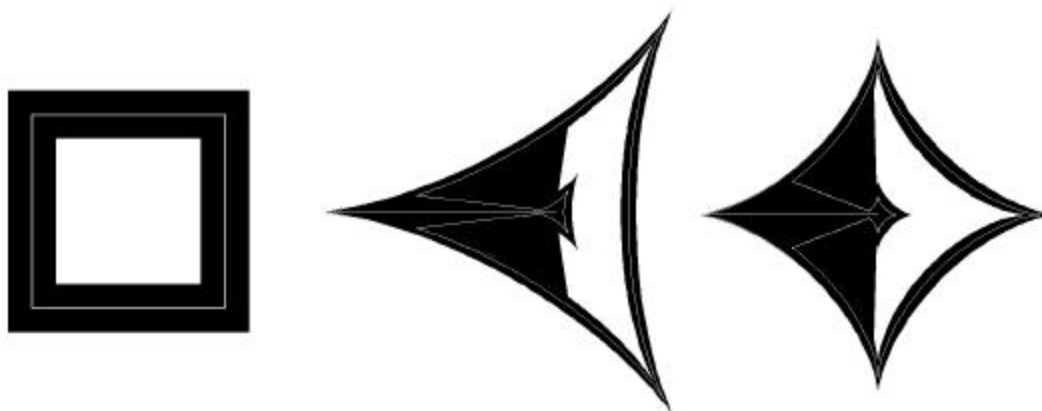
Figure 12: *Hypocycloid spirolateral 1_{90}*
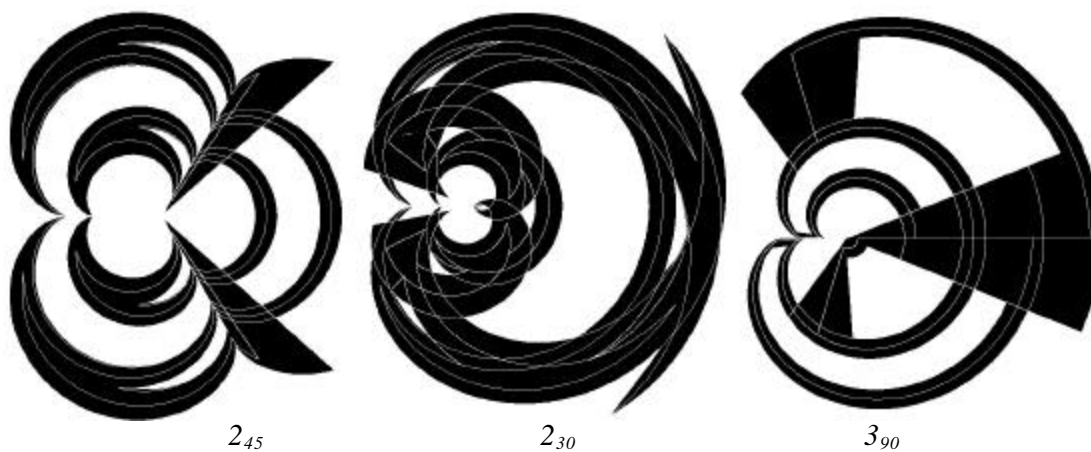
Figure 13: *Epicycloid spirolateral 1_{90}*

Figures 14 and 15 display sample of spirolaterals using the Hypocycloid and Epicycloid curves.

1_{60} $\qquad\qquad\qquad$ 2_{45} $\qquad\qquad\qquad$ 3_{90}

Figure 14: *Hypocycloid spirolaterals*

2_{45} $\qquad\qquad\qquad$ 2_{30} $\qquad\qquad\qquad$ 3_{90}

Figure 15: *Epicycloid spirolaterals*

These transformations produce much more delicate curves due to the effect of the line thickness. The Hypocycloid is particularly interesting because of the very sharp point where its curves meet. It is also the transformation that produces concave curves. The Epicycloid is somewhat similar to the antiMercator and Circular transformation except for the unequal treatment of vertical and horizontal lines, as they generate a changing radius. Both generate quite a series of unexpected results.

6. Observations

As the mathematical basis of these forms evolves and image variations are developed, there is a sense that even though it is at times difficult to predict an outcome, the resulting images have a strange familiarity to them, reminiscent of decorative patterns from the ancient world. On the other hand the sharp edges, the lack of color, and the natural mathematical presentation give them a distinctly futuristic look. It may be very difficult to judge totally new forms since the aesthetic we use is based on all of our previous visual experience and training. The potential for the unexpected is clearly there, even from the simplest of beginnings.

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