Computational math: Assignment 2

Thanks Ting Gao’s support for this HW solutions.

5.2 Using the SVD, prove that any matrix an $\mathbb{C}^{m \times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m \times n}$. Use the 2-norm for your proof.

Proof. Let the SVD of an arbitrary matrix $A_{m \times n}$ is

$$A = U \Sigma V^*$$

where $U_{m \times m}$ and $V_{n \times n}$ are unitary matrices.

Suppose that $m \geq n$.

Denote the singular values of $A$ to be $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Construct a sequence of matrices $\{A_k\}_{k=1}^{\infty}$ as follows:

$$A_k = U \Sigma_k V^*$$

where

$$\Sigma_k = \begin{bmatrix} \sigma_1 + \frac{1}{k} & & \\ & \sigma_2 + \frac{1}{k} & \\ & & \ddots \\ & & & \sigma_n + \frac{1}{k} \end{bmatrix} \in \mathbb{C}^{m \times n}.$$

It’s obvious to see that for any $k \in \mathbb{N}$, we have

$$\text{rank}(A_k) = \text{rank}(U \Sigma_k V^*) = \text{rank}(\Sigma_k) = n.$$

Hence, $\{A_k\}_{k=1}^{\infty}$ are a set of full-rank matrices.

Since

$$\|A - A_k\|_2 = \|U(\Sigma - \Sigma_k)V^*\|_2 = \|\Sigma - \Sigma_k\|_2 = \sqrt{\rho((\Sigma - \Sigma_k)^*(\Sigma - \Sigma_k))} = \frac{1}{k}.$$

We have, $\|A - A_k\|_2 \to 0$ as $k \to 0$, which implies that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m \times n}$. $\square$
5.3 Consider the matrix

\[ A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}. \]

(a) Determine, on paper, a real SVD of \( A \) in the form \( A = U \Sigma V^T \). The SVD is not unique, so find the one that has the minimal number of minus signs in \( U \) and \( V \).

(b) List the singular values, left singular vectors, and right singular vectors of \( A \). Draw a careful, labeled picture of the unit ball in \( \mathbb{R}^2 \) and its image under \( A \), together with the singular vectors, with the coordinates of their vertices marked.

(c) What are the \( 1-, \ 2-, \ \infty- \) and Frobenius norms of \( A \)?

(d) Find \( A^{-1} \) not directly, but via the SVD?

(e) Find the eigenvalues of \( \lambda_1, \lambda_2 \) of \( A \).

(f) Verify that \( \det A = \lambda_1 \lambda_2 \) and \( |\det A| = \sigma_1 \sigma_2 \).

(g) What is the area of the ellipsoid onto which \( A \) maps the unit ball of \( \mathbb{R}^2 \)?

**Solution:**

(a) 

\[ A^*A = \begin{bmatrix} -2 & 11 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}. \]

Since \( A^*A = V(\Sigma^*\Sigma)V^* \) and the eigenvalues of \( A^*A \) are 200, 50, the singular values of \( A \) are

\[ \sigma_1 = 10\sqrt{2}, \ \sigma_2 = 5\sqrt{2}. \]

To find \( U \) and \( V \), we need to calculate the eigenvectors of \( AA^* \) and \( A^*A \). Hence,

\[ U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad V = \begin{bmatrix} -\frac{3}{4} & \frac{1}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}. \]

Therefore, the real SVD of \( A \) with minimal number of minus signs in \( U \) and \( V \) is

\[ A = U \Sigma V^* = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}. \]

(b) The singular values of \( A \) are \( \sigma_1 = 10\sqrt{2}, \ \sigma_2 = 5\sqrt{2} \).

The left singular vectors of \( A \) are

\[ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}. \]
The right singular vectors of $A$ are
\[
\begin{bmatrix}
-\frac{3}{4} \\
\frac{1}{4}
\end{bmatrix}, \begin{bmatrix}
0 \\
1
\end{bmatrix}
\] (c)
\[
\|A\|_1 = \max_{1 \leq j \leq 2} \sum_{i=1}^{2} |a_{ij}| = \max\{12, 16\} = 16.
\]
\[
\|A\|_2 = \sqrt{\rho(A^*A)} = 10\sqrt{2}.
\]
\[
\|A\|_{\infty} = \max_{1 \leq i \leq 2} \sum_{j=1}^{2} |a_{ij}| = \max\{13, 15\} = 15.
\]
\[
\|A\|_F = \sqrt{\text{tr}(A^*A)} = \sqrt{250} = 5\sqrt{10}.
\]
(d) $A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^* = \begin{bmatrix}
0.05 & -0.11 \\
0.1 & -0.02
\end{bmatrix}$.
(e) The eigenvalues of $A$ are $\lambda_1 = \frac{3 + \sqrt{391}}{2}$ and $\lambda_2 = \frac{3 - \sqrt{391}}{2}$.

(f) To find $\lambda_1$ and $\lambda_2$, let’s suppose that $\det(\lambda I - A) = 0$. That is to say,
\[
\lambda^2 - \text{trace}(A)\lambda + \det A = 0.
\]
Hence, it is easy to see that $\lambda_1 \lambda_2 = \det A$.
For arbitrary unitary matrix $U$, we have $\det(UU^*) = \det U \cdot \det U^* = 1$, hence $\det U = \det U^* = \pm 1$. Therefore, we have
\[
(\det A)^2 = \det A^* \cdot \det A = \det(A^*A) = \det(V\Sigma^*\Sigma V^*) = \det V \cdot \det(\Sigma^*\Sigma) \cdot \det V^*.
\]
Thus,
\[
|\det A| = \sqrt{\det(\Sigma^*\Sigma)} = \sqrt{\sigma_1^2 \sigma_2^2} = \sigma_1 \sigma_2.
\]
(g) The area of the ellipsoid is
\[
\pi \sigma_1 \sigma_2 = \pi \cdot 10\sqrt{2} \cdot 5\sqrt{2} = 100\pi.
\]
6.4 Consider the matrix
\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
What is the orthogonal projector \( P \) onto \( \text{range}(A) \), and what is the image under \( P \) of the vector \((1, 2, 3)^*\)?

**Solution:**
The orthogonal projector \( P \) onto \( \text{range}(A) \) is
\[
P = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.
\]
Hence,
\[
P = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.
\]

### 6.5
Let \( P \in \mathbb{C}^{m \times m} \) be a nonzero projector. Show that \( \| P \|_2 \geq 1 \), with equality if and only if \( P \) is an orthogonal projector.

**Proof.** Since \( P \) is a nonzero projector, we have \( P = P^2 \) and \( \| P \|_2 \neq 0 \). Then, based on Cauchy-Schwarz inequality, we have
\[
\| P \|_2 = \| P^2 \|_2 \leq \| P \|_2^2.
\]
Hence, \( \| P \|_2 \geq 1 \).

If \( P \) is an orthogonal projector, then \( P^* = P \). Suppose \( P \) has the SVD of the form \( P = U\Sigma V^* \), where \( UU^* = VV^* = I \).

Hence,
\[
\| P \|_2 = \| P^2 \|_2 = \| PP^* \|_2 = \| \Sigma \Sigma^* \|_2 = \sigma_1^2,
\]
where \( \sigma_1 \) is the largest singular value of \( \Sigma \).

Since \( \| P \|_2 = \| \Sigma \|_2 = \sigma_1 > 0 \). We have \( \sigma_1^2 = \sigma_1 \). Therefore, \( \sigma_1 = 1 \). i.e., \( \| P \|_2 = 1 \).

Assume that the projector \( P \) is not orthogonal. i.e., \( \text{range}(P) \) is not perpendicular to \( \text{range}(I - P) \). Then, we can find a vector \( a \) such that \( Pa \neq a \) and \( a \perp \text{range}(I - P) \).

Hence,
\[
\| Pa \|_2 = \| a + (P - I)a \|_2 > \| a \|_2.
\]
Therefore,
\[
\| P \|_2 = \sup_{\| a \|_2 = 1} \| Pa \|_2 > \sup_{\| a \|_2 = 1} \| a \|_2 = 1.
\]
7.1 Consider matrix

\[ B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

Using any method you like, determine reduced and full QR factorizations \( B = \hat{Q}\hat{R} \) and \( B = QR \).

Solution: Rewrite \( B \) as \( B = [b_1 \ b_2] \), where

\[ b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \]

which are linearly independent. Hence,

\[ q_1 = \frac{b_1}{\|b_1\|} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}, \]

\[ q_2 = \frac{b_2 - (q_1^*b_2)q_1}{\|b_2 - (q_1^*b_2)q_1\|} = \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ -\sqrt{3}/3 \end{bmatrix}. \]

Hence,

\[ B = \hat{Q}\hat{R} = [q_1 \ q_2] \begin{bmatrix} \|b_1\| & q_1^*b_2 \\ 0 & \|b_2 - (q_1^*b_2)q_1\| \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 & \sqrt{6}/6 \\ 0 & \sqrt{3}/3 & -\sqrt{6}/6 \\ \sqrt{2}/2 & \sqrt{3}/3 & -\sqrt{6}/6 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}. \]

When finding \( Q \), we need to find another vector \( q_3 \) satisfying \( q_1^*q_3 = 0, q_2^*q_3 = 0 \) and \( \|q_3\| = 1 \). Hence, we have \( q_3 = [\sqrt{\frac{\pi}{6}} - \sqrt{\frac{\pi}{6}} - \sqrt{\frac{\pi}{6}}]^T \). Therefore, the full QR factorization is

\[ B = QR = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 & \sqrt{6}/6 \\ 0 & \sqrt{3}/3 & -\sqrt{6}/6 \\ \sqrt{2}/2 & \sqrt{3}/3 & -\sqrt{6}/6 \end{bmatrix}. \]

7.5 Let \( A \) be an \( m \times n \) matrix \( (m \geq n) \), and let \( A = \hat{Q}\hat{R} \) be a reduced QR factorization.

(a) Show that \( A \) has rank \( n \) if and only if all the diagonal entries of \( \hat{R} \) are nonzero.

(b) Suppose \( \hat{R} \) has \( k \) nonzero diagonal entries for some \( k \) with \( 0 \leq k \leq n \). What does this imply about the rank of \( A \)?
Proof. Let’s first denote that

\[ A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}. \]

Using the induction method to show the proof of (a) and (b), we only need to show that if \( r_{kk} \neq 0 \) then

\[ \text{rank}(A_k) \geq \text{rank}(A_{k-1}) + 1, \]

where

\[ A_k := (a_1|\cdots|a_k). \]

Combing the formula

\[ a_k = \sum_{j=1}^{k-1} r_{jk}q_j + r_{kk}q_k, \quad k = 1, \ldots, n, \]

we have

\[ a_k \in \text{span}\{q_1, \cdots, q_{k-1}, q_k\}, \]

but

\[ a_1, \cdots, a_{k-1} \in \text{span}\{q_1, \cdots, q_{k-1}\}. \]

It implies that

\[ a_k \notin \text{span}\{a_1, \cdots, a_{k-1}\}. \]

Thus

\[ \text{rank}(A_k) \geq \text{rank}(A_{k-1}) + 1. \]

Therefore

\[ \text{rank}(A) = n, \]

when all the diagonal entries of \( \hat{R} \) are nonzero, and

\[ \text{rank}(A) \geq k, \]

when \( \hat{R} \) has \( k \) nonzero diagonal entries.

\[ \square \]

8 Prove \( P_j = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1}, \quad j = 2, 3, \ldots, n. \)
**Proof.** Since

\[
P_j = I - \begin{bmatrix} q_1 & \cdots & q_{j-1} \end{bmatrix} \begin{bmatrix} q_1^* \\ \vdots \\ q_{j-1}^* \end{bmatrix} = I - \sum_{i=1}^{j-1} q_i q_i^*.
\]

and

\[
\prod_{i=1}^{j-1} P_{q_i} = \prod_{i=1}^{j-1} (I - q_i q_i^*) = I - q_{j-1} q_{j-1}^* - \cdots - q_1 q_1^*
\]

for \( q_i \perp q_j (i \neq j) \), i.e., \( q_i q_i^* q_j q_j^* = 0 (i \neq j) \) (This is a zero matrix). Therefore,

\[
P_j = P_{q_{j-1}} \cdots P_{q_2} P_{q_1}.
\]