Approximating \( \ln 2 \) with Riemann Sums

To approximate the number \( \ln 2 \), we consider upper and lower Riemann sums. By definition,

\[
\ln 2 = \int_1^2 \frac{1}{t} \, dt
\]

In Mathematica, we define \( f(t) = \frac{1}{t} \).

\[
\text{In[1]:= } f[t_] := 1/t
\]

Upon partitioning the interval \( I = [1, 2] \) into \( n \) subintervals, each of length \( \frac{1}{n} \), we may form \( L(n) \), the left hand Riemann sum for \( f \) over \( I \):

\[
\text{In[2]:= } L[n_] := \text{Sum}[f[1 + k/n] / n, \{k, 0, n - 1\}]
\]

To see the form of the summation in the above definition we input the following instruction:

\[
\text{In[3]:= TraditionalForm[HoldForm[Sum[f[1 + k/n] / n, \{k, 0, n - 1\}]]]}
\]

\[
\text{Out[3]/TraditionalForm=}
\sum_{k=0}^{n-1} \frac{f\left(1 + \frac{k}{n}\right)}{n}
\]

Thus, we have defined \( L \) as:

\[
L(n) = \sum_{k=0}^{n-1} \frac{f\left(1 + \frac{k}{n}\right)}{n}.
\]

Geometrically, we see in the figure above that

\[
L(n) = A_{1,n} + A_{2,n} + A_{3,n} + \ldots + A_{n,n}.
\]
Note that \( L(1) \geq L(2) \geq L(3) \geq \ldots \geq \ln 2 \) since \( f \) is decreasing on \( I \).

Now we consider the right hand Riemann sum \( R(n) \) for \( f \) on \( I \), assuming a uniform partition as before.

\[
\ln[4] = R[n] = \text{Sum}[f[1 + k/n] / n, \{k, 1, n\}]
\]

As before, to check our formula defining \( R \), we input:

\[
\ln[5] = \text{TraditionalForm}[\text{HoldForm}[\text{Sum}[f[1 + k/n] / n, \{k, 1, n\}]]]
\]

\[
\text{Out}[5] = \text{TraditionalForm}
\sum_{k=1}^{n} \frac{f(1 + \frac{k}{n})}{n}
\]

So we have that
\[
R(n) = \sum_{k=1}^{n} \frac{f(1 + \frac{k}{n})}{n}.
\]

Geometrically, it is evident from the figure that
\[
R(n) = a_{1,n} + a_{2,n} + a_{3,n} + \ldots + a_{n,n}
\]

Note that \( L(1) \leq R(2) \leq R(3) \leq \ldots \leq \ln 2 \) since \( f \) is decreasing on \( I \).

Claim: The value of \( \ln 2 \) is 0.69 rounded to the hundredths place.

To see this, consider the following computation:
\textbf{In[6]:= \hspace{0.5cm}} \texttt{n = 1;}
\texttt{While} [\texttt{L[n]} - \texttt{R[n]} > 0.003, \\
\hspace{1cm} \texttt{n = n + 1}; \\
\texttt{n}]
\textbf{Out[8]= \hspace{0.5cm}} 167

Now observe that

\textbf{In[9]:= \hspace{0.5cm}} \texttt{N[\texttt{L[167]}]} \texttt{=} \texttt{L(167)} \hspace{0.5cm}
\texttt{N[\texttt{R[167]}]} \texttt{=} \texttt{R(167)}
\textbf{Out[9]= \hspace{0.5cm}} 0.694646 \texttt{=} \texttt{L(167)}
\textbf{Out[10]= \hspace{0.5cm}} 0.691652 \texttt{=} \texttt{R(167)}

and since \hspace{0.5cm} \texttt{R(n)} \leq \texttt{ln 2} \leq \texttt{L(n)} \hspace{0.5cm} \text{ for any positive integer } \texttt{n}, \hspace{0.5cm} \text{we are certain that}

\hspace{0.5cm} \texttt{0.691652} \leq \texttt{ln 2} \leq \texttt{0.694646}.

Therefore, we conclude that \hspace{0.5cm} \texttt{ln 2 = 0.69} \hspace{0.5cm} \text{accurately rounded to the hundredths place.}