Points of Nondifferentiability

The function \( f(x, y) = \sqrt{|x|} - \frac{2|xy|}{\sqrt{x^2 + y^2}} \) is nondifferentiable at \((0, 0)\), although \( f_x(0, 0) = f_y(0, 0) = 0 \).

Observe that \( f(x, 0) = x^2 \), so \( f_x(0, 0) \) is the slope of the parabola \( z = x^2 \) at the origin of the \( xz \) plane ==>
\( f_x(0, 0) = 0 \).

Similarly, \( f(0, y) = y^2 \), so \( f_y(0, 0) \) is the slope of the parabola \( z = y^2 \) at the origin of the \( yz \) plane ==>
\( f_y(0, 0) = 0 \).

However, that the trace of the surface \( z = f(x, y) \) in the plane \( y = x \) has a corner at the origin.
This trace is the graph of \( z = u \) in the \( uv \) plane, where the \( u \) axis coincides with the line \( \{(x, y, z) : x = y, z = 0\} \).
Therefore, the surface \( z = f(x, y) \) has no tangent plane at the origin.
Hence, the function \( f \) is not differentiable at \((0, 0)\).

It is also the case that the trace of the surface \( z = f(x, y) \) in the plane \( y = -x \) has a corner at the origin.
This trace is the graph of \( z = v \) in the \( vz \) plane, where the \( v \) axis coincides with the line \( \{(x, y, z) : x = -y, z = 0\} \).

Exercises:
1. Show that both limits: \( \lim_{x \to 0^+} f_x(x, y) \) and \( \lim_{x \to 0^-} f_x(x, y) \) exist but are distinct whenever \( y \neq 0, \pm 1 \).

   Consequently, \( f_y(0, y) \) does not exist whenever \( y \neq 0, \pm 1 \) ==>
\( f \) is not differentiable at any of these points along the \( y \) axis.

2. Show that \( f_y(1, 0) = f_y(0, -1) = 0 \).

3. Show that both \( \lim_{y \to 0^+} f_y(x, y) \) and \( \lim_{y \to 0^-} f_y(x, y) \) exist but are distinct whenever \( x \neq 0, \pm 1 \).

   Consequently, \( f_x(x, 0) \) does not exist whenever \( x \neq 0, \pm 1 \) ==>
\( f \) is not differentiable at any of these points along the \( x \) axis.

4. Show that \( f_y(1, 0) = f_y(1, -1) = 0 \).

5. Examine \( f \) for differentiability at the point \( P = (1, 0) \).

   Recall: \( f \) differentiable at \( P \) \iff \( f(1 + \Delta x, \Delta y) - f(1, 0) = f(1, 0) \Delta x + f_y(1, 0) \Delta y + e_1 \Delta x + e_2 \Delta y \)
where \( e_1 \) and \( e_2 \) approach 0 as both \( \Delta x \to 0 \) and \( \Delta y \to 0 \).

We graph the surface \( z = f(x, y) \) below.
In order to see more clearly the indentations in the graph of \( z = f(x,y) \) we plot a contour map for this function.

\[
\begin{align*}
> \text{l:=line([0,-7],[0,7]):m:=line([-7,0],[7,0]):} \\
> \text{cmap:=implicitplot}\{f(x,y)=8,f(x,y)=6,f(x,y)=4,f(x,y)=2,f(x,y)=1,f(x,y)=1/2}\},x=-7..7,y=-7..7,scaling=constrained,grid=[80,80],axesfont=[\text{TIMES,ITALIC,10}],\text{tickmarks=[3,3],color=navy,thickness=2):} \\
> \text{display(l,m,cmap);}
\end{align*}
\]
The plot of the space curves along the graph of the surface $z = f(x, y)$ in the planes $y = \frac{1}{2}$, $y = 1$ and $y = 2$ shown below clearly indicate that $f_y(0, \frac{1}{2})$ and $f_y(0, 2)$ do not exist but $f_y(0, 1) = 0$.

\[
\begin{align*}
&\text{> s5:=spacecurve([t,1,f(t,1)],t=-1.8..1.8,color=red,thickness=4):} \\
&\text{> l5:=line([0,1,0],[0,1,f(0,1)],linestyle=2,color=red):} \\
&\text{> s6:=spacecurve([t,2,f(t,2)],t=-1.8..1.8,color=green,thickness=4):} \\
&\text{> l6:=line([0,2,0],[0,2,f(0,2)],linestyle=2,color=green):} \\
&\text{> s7:=spacecurve([t,1/2,f(t,1/2)],t=-1.8..1.8,color=blue,thickness=4,axesfont=[\text{TIMES,ITALIC},8],orientation=[27,67]):} \\
&\text{> l7:=line([0,1/2,0],[0,1/2,f(0,1/2)],linestyle=2,color=blue):} \\
&\text{display(s5,l5,s6,l6,s7,l7,t1,t2,t3,axes=normal);}
\end{align*}
\]