General Solution to a Linear Homogeneous First Order System of ODEs with Constant Coefficients

Consider the system of DEs:  

\[ X'(t) = A X(t) \]  \( (*) \)

where \( A \) is a real constant \( n \times n \) matrix.

It will have solutions of the form:  

\[ X(t) = e^{\lambda t} K \]

where \( \lambda \) is an eigenvalue of the coefficient matrix \( A \).

A complete description of the \( n \times 1 \) column vector \( K \) is given by the following three theorems.

I. Matrix \( A \) has all Distinct Real Eigenvalues

**Theorem 1**  Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be \( n \) real, distinct eigenvalues of \( A \) and let \( K_1, K_2, \ldots, K_n \) be corresponding eigenvectors. Then the general solution to \( (*) \) on \( (-\infty, \infty) \) is given by:

\[ X(t) = c_1 e^{\lambda_1 t} K_1 + c_2 e^{\lambda_2 t} K_2 + \ldots + c_n e^{\lambda_n t} K_n \]

II. Matrix \( A \) has Complex Eigenvalues

Suppose that \( \lambda = \alpha + i \beta \), \( K = R + i S \) is an eigenpair for the matrix \( A \), where \( R \), \( S \) denote real, constant \( n \times 1 \) vectors. Then a complex-valued solution to \( (*) \) is given by

\[ X(t) = e^{\alpha t} K = e^{\alpha t} (\cos \beta t + i \sin \beta t) R + i \sin \beta t) S \]

\[ = e^{\alpha t} \{ [\cos \beta t] R - [\sin \beta t] S \} + i [\cos \beta t] S + [\sin \beta t] R \} \]

We call \( X_1(t) = e^{\alpha t} [\cos \beta t] R - [\sin \beta t] S \) the **real part** of \( X(t) \) and \( X_2(t) = e^{\alpha t} [\cos \beta t] S + [\sin \beta t] R \) the **imaginary part** of \( X(t) \).

**Lemma**  Let \( X(t) = X_1(t) + i X_2(t) \) be a complex-valued solution to the system \( (*) \). Then both \( X_1(t) \) and \( X_2(t) \) are real-valued solutions of \( (*) \).

**Proof**  Since \( \frac{d}{dt} [X_1(t) + i X_2(t)] = A [X_1(t) + i X_2(t)] \), we have

\[ X_1'(t) + i X_2'(t) = AX_1(t) + i AX_2(t) \]

Now equating the real and imaginary parts of this equality, we find

\[ X_1'(t) = AX_1(t) \quad \text{and} \quad X_2'(t) = AX_2(t) \]

as required.

This discussion helps to motivate the following result.
\textbf{Theorem 2} \ If \ \( \lambda = \alpha + i \beta \) \ is a complex-valued eigenvalue of \( A \) \ and \( K = R + iS \) \ is an associated eigenvector then

\[ X_1(t) = e^{\alpha t} \left[ ( \cos \beta t ) R - ( \sin \beta t ) S \right] \] \quad \text{and} \quad \[ X_2(t) = e^{\alpha t} \left[ ( \cos \beta t ) S + ( \sin \beta t ) R \right] \]

are linearly independent real-valued solutions of \((*)\) on \((-\infty, \infty)\).

\section*{III. Matrix \( A \) \ has Repeated Eigenvalues}

\textbf{Theorem 3}

\textbf{Case 1} \ Suppose the \( n \times n \) matrix \( A \) \ has less than \( n \) distinct eigenvalues, however, there still exists a set of \( n \) linearly independent eigenvectors. Then the general solution to \((*)\) consists of \( n \) solutions of the form \( X(t) = e^{\lambda t} K \) \ where \( \lambda, K \) \ is an eigenpair.

\textbf{Case 2} \ Suppose that \( A \) \ has only \( k < n \) linearly independent eigenvectors. Then

\( \text{(i)} \) the general solution to \((*)\) includes \( k \) solutions of the form

\[ X(t) = e^{\lambda t} \] \ where \( \lambda, K \) \ is an eigenpair.

\( \text{(ii)} \) To find additional solutions, pick an eigenvalue \( \lambda \) and find all vectors \( V \)

such that \( (A - \lambda I)^2 V = 0 \) \ but \( (A - \lambda I) V \neq 0 \).

For each such \( V \) \ it follows that

\[ X(t) = e^{\lambda t} \left[ I + t (A - \lambda I) \right] V \] \ is another solution.

\( \text{(iii)} \) If we still do not have enough solutions, we find all vectors \( V \)

for which \( (A - \lambda I)^3 V = 0 \) \ but \( (A - \lambda I)^2 V \neq 0 \).

For each such \( V \) \ it follows that

\[ X(t) = e^{\lambda t} \left[ I + t (A - \lambda I) + \frac{t^2}{2!} (A - \lambda I)^2 \right] V \] \ is also a solution.

\( \text{(iv)} \) We continue in this manner until we obtain \( n \) linearly independent solutions.

Note: In regards to Case 2 it can be shown that if \( \lambda_o \) \ is an eigenvalue of multiplicity \( k \) \ for matrix \( A \), i.e. \( (\lambda - \lambda_o)^k \) \ is a factor of \( \det(A - \lambda I) \), then there exists an integer \( N \leq k \) \ such that \( (A - \lambda I)^N V = 0 \) \ has at least \( k \) \ linearly independent solutions. Thus, corresponding to the eigenvalue \( \lambda_o \), we can compute \( k \) \ linearly independent solutions to \((*)\) each having the form

\[ X(t) = e^{\lambda_o t} \left[ I + t (A - \lambda_o I) + \frac{t^2}{2!} (A - \lambda_o I)^2 + \cdots + \frac{t^{(N-1)}}{(N-1)!} (A - \lambda_o I)^{(N-1)} \right] V \]