REDUCTION OF A CAUCHY–EULER DE TO CONSTANT COEFFICIENTS

Consider the fourth order homogeneous Cauchy–Euler DE:

\[
ax^4 \frac{d^4y}{dx^4} + bx^3 \frac{d^3y}{dx^3} + cx^2 \frac{d^2y}{dx^2} + dx \frac{dy}{dx} + \varepsilon y = 0. \tag{1}
\]

We may transform (1) into a constant coefficient DE with the substitution:

\[
x = e^t \Rightarrow t = \ln x \quad \text{for} \quad x \in (0, +\infty).
\]

Now we have the representations for the first four derivatives of \(y\) with respect to \(x\):

\[
\begin{align*}
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x} \cdot \frac{dy}{dt} \quad \text{by the Chain Rule.} \\
\frac{d^2y}{dx^2} &= \frac{1}{x} \cdot \frac{d}{dt} \left( \frac{dy}{dt} \right) \cdot \frac{dt}{dx} + \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{1}{x} \right) \quad \text{by the Product and Chain Rules} \\
\frac{d^3y}{dx^3} &= \frac{1}{x^2} \cdot \frac{d}{dt} \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right) \cdot \frac{dt}{dx} + \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{1}{x^2} \right) \quad \text{by the Product and Chain Rules} \\
\frac{d^4y}{dx^4} &= \frac{1}{x^3} \cdot \frac{d}{dt} \left( \frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} \right) \cdot \frac{dt}{dx} + \frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} \cdot \frac{d}{dx} \left( \frac{1}{x^3} \right) \\
\end{align*}
\]

--by the Product and Chain Rules

Upon substitution for \(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\) and \(\frac{d^4y}{dx^4}\) back into (1) we find that

\[
a \left( \frac{d^4y}{dt^4} - 6 \frac{d^3y}{dt^3} + 11 \frac{d^2y}{dt^2} - 6 \frac{dy}{dt} \right) + b \left( \frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} \right) + c \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right) + d \frac{dy}{dt} + \varepsilon y = 0.
\]
Regrouping terms, we have the equivalent constant coefficient DE in terms of the independent variable $t$:

\[
\frac{d^4y}{dt^4} + (-6a + b) \frac{d^3y}{dt^3} + (11a - 3b + c) \frac{d^2y}{dt^2} + (-6a + 2b - c + d) \frac{dy}{dt} + \varepsilon y = 0
\]

(2)

Now to solve for $y$ in (2) we set $y = e^{mt}$ which yields:

\[
[a \ m^4 + (-6a + b) \ m^3 + (11a - 3b + c) \ m^2 + (-6a + 2b - c + d) \ m + \varepsilon y] \ e^{mt} = 0.
\]

Consequently, the characteristic equation corresponding to (2) is:

\[
a \ m^4 + (-6a + b) \ m^3 + (11a - 3b + c) \ m^2 + (-6a + 2b - c + d) \ m + \varepsilon y = 0 \quad \star
\]

We determine several solutions for (1) based on the roots to (\star).

**Case 1: Distinct Real Roots**

In case (\star) factors as:

\[
(m - m_1)(m - m_2)(m - m_3)(m - m_4) = 0
\]

where $m_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$ then

$m = m_1, m_2, m_3, m_4$ are all distinct real roots to (\star).

So

\[
y = c_1 e^{m_1 t} + c_2 e^{m_2 t} + c_3 e^{m_3 t} + c_4 e^{m_4 t}
\]

is the solution to (2). Since $t = \ln x$ and $e^{\ln x} = x$ we find

\[
y = c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3} + c_4 x^{m_4}
\]

to be the general solution to (1) for $x \in (0, +\infty)$.

**Case 2: A Repeated Real Root of Multiplicity Four**

In case (\star) factors as:

\[
(m - m_0)^4 = 0
\]

then $m = m_0$ is a root of multiplicity four to (\star).

So

\[
y = c_1 e^{m_0 t} + c_2 t e^{m_0 t} + c_3 t^2 e^{m_0 t} + c_4 t^3 e^{m_0 t}
\]

is the solution to (2). Upon substituting $t = \ln x$ we find

\[
y = c_1 x^{m_0} + c_2 (\ln x) x^{m_0} + c_3 (\ln x)^2 x^{m_0} + c_4 (\ln x)^3 x^{m_0}
\]

to be the general solution to (1) for $x \in (0, +\infty)$.

**Case 3: A Repeated Pair of Conjugate Roots**

In case (\star) factors as:

\[
(m - (\alpha + i \beta))^2 (m - (\alpha - i \beta))^2 = 0
\]

then $m = \alpha \pm i \beta$ is a repeated pair of conjugate roots to (\star).

So

\[
y = e^{\alpha t} (c_1 + c_2 t) \cos(\beta t) + e^{\alpha t} (c_3 + c_4 t) \sin(\beta t)
\]

is the solution to (2). Then setting $t = \ln x$ we find
$y = x^\alpha \left( c_1 + c_2 \ln x \right) \cos(\beta \ln x) + x^\alpha \left( c_3 + c_4 \ln x \right) \sin(\beta \ln x)$

to be the general solution to (1) for $x \in (0, +\infty)$.

It is important to recognize that upon setting $y = x^m$ in (1) yields:

$$[a m \left( m - 1 \right) \left( m - 2 \right) \left( m - 3 \right) + b m \left( m - 1 \right) \left( m - 2 \right) + c m \left( m - 1 \right) + d m + \varepsilon ] x^m = 0$$

That is,

$$[a m^4 + (-6a + b) m^3 + (11a - 3b + c) m^2 + (-6a + 2b - c + d) m + \varepsilon y ] x^m = 0.$$

Thus, $y = x^m$ is a solution to (1) $\iff$ $m$ is a root of:

$$a m^4 + (-6a + b) m^3 + (11a - 3b + c) m^2 + (-6a + 2b - c + d) m + \varepsilon y = 0 \quad (3)$$

Observe that (3) agrees precisely with the characteristic equation (•) for (2).

In general, the $n^{th}$ order C-E DE:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = 0 \quad (4)$$

when transformed to the constant coefficient DE:

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + A_1 \frac{dy}{dt} + A_0 y = 0$$

by setting $x = e^t$ will possess the characteristic equation:

$$\alpha_n m^n + \alpha_{n-1} m^{n-1} + \ldots + \alpha_1 m + \alpha_0 = 0$$

if and only if the equation having precisely the same coefficients for $m$:

$$[\alpha_n m^n + \alpha_{n-1} m^{n-1} + \ldots + \alpha_1 m + \alpha_0 ] x^m = 0$$

is obtained directly from (4) by setting $y = x^m$. 

\[\boxed{\text{}}\]