Maple and Phase Portraits

We may generate the phase portrait of a system of nonlinear first order DEs using Maple.

For the system

\[
\begin{align*}
\frac{dx}{dt} &= 2 - 4x - 15y \\
\frac{dy}{dt} &= 4 - x^2
\end{align*}
\]

we will identify the critical points, and then plot several trajectories and the related slope field, by utilizing Maple's `plots`, `plottools` and `DEtools` packages.

First, we must open the appropriate Maple packages.

```maple
> with(plots): with(plottools): with(DEtools):
```

Now we find the critical points for the system, using the `solve` command.

```maple
> solve({2-4*x-15*y=0,4-x^2=0},{x,y});
```

\[
\begin{align*}
&\{x = 2, y = \frac{-2}{3}\}, \{x = -2, y = \frac{2}{3}\}
\end{align*}
\]

Next, we specify a set of points through which we would like to have trajectories pass through in the phase portrait for the system. To generate a large set of initial points quickly, we use a "composite" sequence command.

```maple
> initialset:={seq(seq([x(0)=a,y(0)=b],a=-2..2),b=-2..2)};
```

```maple
initialset := {\[x(0) = -2, y(0) = -2\], \[x(0) = -2, y(0) = -1\], \[x(0) = -2, y(0) = 0\], \[x(0) = -2, y(0) = 1\], \[x(0) = -1, y(0) = 2\], \[x(0) = 0, y(0) = 0\], \[x(0) = 0, y(0) = 1\], \[x(0) = 1, y(0) = 2\], \[x(0) = 2, y(0) = 0\], \[x(0) = 2, y(0) = 1\], \[x(0) = 2, y(0) = 2\]}
```

Now let's assign the name `A` to the plot of the trajectories through the initial set of points defined above embedded in the system's slope field.

```maple
> A:=DEplot([diff(x(t),t)=2-4*x(t)-15*y(t),diff(y(t),t)=4-x(t)^2],[x(t),y(t)],t=-3.3,x=-8..6,y=-5..7,initialset,stepsize=0.01,color=blue,linecolor=magenta,arrows=medium,axes=boxed):
```

In order to display the critical points in our phase portrait, we will blacken-in ellipses centered at each critical solution.

```maple
> P1:=ellipse([2,-2/5],0.1,0.15,filled=true,color=black):
> P2:=ellipse([-2,2/3],0.1,0.15,filled=true,color=black):
```

Finally, we display all these characteristic features of our system in a single phase portrait.

```maple
> display([P2,A,P1]);
```
Observe that the critical point, \( P_2 = (\frac{-2}{3}, \frac{2}{3}) \), is asymptotically stable, since every trajectory that starts sufficiently close to \( P_2 \) approaches \( P_2 \) as \( t \to \infty \). Thus, \( P_2 \) is a spiral sink.

The critical point \( P_1 = (2, \frac{-2}{5}) \) is unstable. It may be classified as a saddle point for the system.

We may verify our classifications of these critical points analytically.

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(i) Observe that the substitution \( u = x - 2, \ v = y + \frac{2}{5} \) in (1) and (2) yields the almost linear system:

\[
\begin{align*}
\frac{du}{dt} &= -4u - 15v \\
\frac{dv}{dt} &= -4u - v^2
\end{align*}
\]

having \( (0, 0) \) as the critical point corresponding to \( P_1 \). The associated linear system:

\[
\begin{align*}
\frac{du}{dt} &= -4u - 15v \\
\frac{dv}{dt} &= -4u
\end{align*}
\]

has characteristic equation: \( \lambda (\lambda + 4) - 60 = 0 = (\lambda - 6)(\lambda + 10) \). Since the roots:

\( \lambda = 6 \) and \( \lambda = -10 \) are real and unequal, it follows that \( P_1 \) is a saddle point for the system (1)&(2).

(ii) The substitution \( p = x + 2, \ q = y - \frac{2}{3} \) in (1) and (2) yields the almost linear system:
\[
\frac{dp}{dt} = -4p - 15q \\
\frac{dq}{dt} = 4p - p^2
\]

having \((0, 0)\) as the critical point corresponding to \(P_2\). The associated linear system:

\[
\frac{dp}{dt} = -4p - 15q \\
\frac{dq}{dt} = 4p
\]

has characteristic equation: \(\lambda (\lambda + 4) + 60 = 0 = (\lambda - 6)(\lambda + 10)\). Since the roots:
\[
\lambda_1 = -2 + \frac{\sqrt{326}}{2} i \quad \text{and} \quad \lambda_2 = -2 - \frac{\sqrt{326}}{2} i
\]
are complex conjugates, it follows that \(P_2\) that is a \textit{spiral point}
for the system (1)&(2). Moreover, because \(\text{Re}(\lambda_1) = -2 = \text{Re}(\lambda_2)\) are both negative, \(P_2\) is \textit{asymptotically stable}.

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**Extra Credit Assignment**

In order to receive additional points for submitting this assignment, you must include comments in your report to fully support your reasoning regarding both the mathematical and the programming aspects of your solutions.

(All comments should be expressed in complete sentences.)

For each of the problems 1 - 6, do the following:

\(\text{(a)}\) Use \textit{Maple} to find the critical points of the given autonomous system.

\(\text{(b)}\) Sketch a phase portrait and slope field over the region \(-5 < x < 5, -5 < y < 5\) with \textit{Maple}.

Be certain to display all of its characteristic features as in the example above.

\(\text{(c)}\) Based on the qualitative nature of the phase portrait, classify each critical point as being either a node, a saddle point, a center, or a spiral point.

If the critical point is a node, is it proper or improper? Is it a nodal sink or a nodal source?

Also classify each critical point as being stable, asymptotically stable, or unstable.

Explain your reasoning in each case.

1. \[
\frac{dx}{dt} = x - y \\
\frac{dy}{dt} = x + 3y - 4
\]
2. \[
\frac{dx}{dt} = 2x - y \\
\frac{dy}{dt} = x - 3y
\]
3. \[
\frac{dx}{dt} = x - 2y + 3 \\
\frac{dy}{dt} = x - y + 2
\]
4. \[
\frac{dx}{dt} = 1 - y^2 \\
\frac{dy}{dt} = x + 2y
\]
5. \[
\frac{dx}{dt} = x - 2y \\
\frac{dy}{dt} = 4x - x^3
\]
6. \[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = x - x^3
\]

7. \(\text{(a)}\) Find the trajectories for the system in problem 6 analytically (i.e., by hand).

You your answer should consist of a family of equations of the form \(F(x, y) = C\).

For this, first rewrite the system as a single first order DE, and then separate variables.

\(\text{Hint:} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \).

\(\text{(b)}\) Determine all of the critical points for the problem 6 system analytically.

\(\text{(c)}\) Describe the behavior of \((x(t), y(t))\) as \(t \to \infty\).

8. \(\text{(a)}\) Use the table in \textit{Figure 7.17 and Theorem 1: On the Stability of Linear Systems} in the handout to classify the type and stability of the critical point of the linear system in problem 2 above.

\(\text{(b)}\) Use the table in \textit{Figure 7.17 and Theorem 2: On the Stability of Almost Linear Systems} in the handout to classify the type and stability of each of the critical points of the almost linear system in problem 4 above.