A fractal is a mathematical set that exhibits a repeating pattern displayed at every scale. It is also known as expanding symmetry or evolving symmetry. If the replication is exactly the same at every scale, it is called a self-similar pattern. An example of this is the Menger Sponge. Fractals can also be nearly the same at different levels. This latter pattern is illustrated in small magnifications of the Mandelbrot set. Fractals are different from other geometric figures because of the way in which they scale. As mathematical equations, fractals are usually nowhere differentiable. The term fractal was first used by mathematician Benoît Mandelbrot in 1975. He based it on the Latin frāctus meaning broken or fractured. Fractal patterns with various degrees of self-similarity can be found in nature, technology and art.

Koch Snowflake

The Koch snowflake (also known as the Koch star and Koch island) is a mathematical curve and one of the earliest fractals to have been described. The Koch snowflake is based on the Koch curve, which appeared in a 1904 paper titled "On a continuous curve without tangents, constructible from elementary geometry" by the Swedish mathematician Helge von Koch.

Construction

The Koch curve can be constructed by starting with an equilateral triangle, then recursively altering each line segment that forms a side of the figure as follows:
1. divide the line segment into three segments of equal length.
2. draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
3. remove the line segment that is the base of the triangle from step 2.
4. After one iteration of this process, the result is a shape similar to the Star of David.
5. The Koch curve is the limit approached as the above steps are followed over and over again.

![Initial equilateral triangle](image1)

<table>
<thead>
<tr>
<th>Initial equilateral triangle</th>
<th>1st iteration</th>
<th>2nd iteration</th>
</tr>
</thead>
</table>

![3rd iteration](image2)

<table>
<thead>
<tr>
<th>3rd iteration</th>
<th>6th iteration</th>
</tr>
</thead>
</table>
Properties

1. The Koch curve has **infinite length** because each time the above steps are applied to each line segment of the figure then the resulting figure will possess four times as many line segments as before. The length of each new segment being $(1/3)^d$ the length of each segment from the previous stage. Hence, the total length of the figure increases by $(4/3)^d$ at each iteration. Thus, the length of the figure at the $n^{th}$ iteration will be $(4/3)^n$ times the perimeter of the original triangle.

   As $n \to +\infty$ then $(4/3)^n \to +\infty$ as well, so the limiting curve has infinite perimeter.

2. The Koch curve is continuous everywhere but differentiable nowhere.

3. As noted above, when any line segment of the Koch curve is scaled down, it is replaced by: 

   $N = 4$ line segments, each being $r = 1/3$ the length of the previous one.

   We call $r$ the **scaling factor** for this curve, and by definition, the **fractal dimension** of the Koch curve is $D$ where:

   $$\left(\frac{1}{r}\right)^D = N \iff 3^D = 4 \iff D \cdot \ln 3 = \ln 4 \iff D = \frac{\ln 4}{\ln 3} \approx 1.262.$$ 

   (the higher the fractal dimension - the greater the degree of complexity of the curve)

4. Taking $s$ as the side length of the initial equilateral triangle, then the original triangle’s area is:

   $$A_0 = \frac{s^2 \sqrt{3}}{4}.$$ 

   At every successive iterate, recall that the side length of each newly constructed triangle is $(1/3)^d$ the length of a side from the previous iteration. Since the area of each triangle constructed at the $n^{th}$ step is proportional to the square of its side length, then the area of each triangle constructed at that step must be $(1/9)^d$ the area of each triangle constructed at the $(n-1)^{th}$ step. In each iteration after the first, 4 times as many triangles are constructed than in the previous iteration, and because the first iteration adds 3 triangles, then the $n^{th}$ iteration must add a total of $3 \cdot 4^{(n-1)}$ triangles. Combining these two formulas yields the recursive area formula:

   $$A_n = A_{n-1} + \frac{3 \cdot 4^{n-1}}{9^n} A_0, \; n \geq 1,$$ 

   where $A_0$ is area of the original triangle.

Substituting in $A_0 = \frac{4}{3} A_0$, and expanding yields:

$$A_n = \frac{4}{3} A_0 + \sum_{k=2}^{n} \left( \frac{3 \cdot 4^{k-1}}{9^k} A_0 \right) = \left( \frac{4}{3} + \frac{1}{3} \sum_{k=2}^{n} \frac{3 \cdot 4^{k-1}}{9^k} \right) A_0$$

In the limit, as $n \to +\infty$, the sum of the powers of $4/9$ in the formula above converges to $4/5$.

Hence, $\lim_{n \to +\infty} A_n = \left( \frac{4}{3} + \frac{1}{3} \frac{4}{5} \right) A_0 = \frac{8}{5} A_0$.

So the area of a Koch snowflake is $8/5$ times the area of the original triangle, or $\frac{2s^2 \sqrt{3}}{5}$ sq units.

Therefore, the **infinite perimeter** of the Koch triangle encloses only a **finite area**.
**Koch Snowflake**

![Diagram of a Koch Snowflake]

Specifications at the $n^{th}$ iterate for $n=0, 1, 2, \ldots$

<table>
<thead>
<tr>
<th>Number of Sides $N_n = 4 \cdot N_{n-1}$</th>
<th>Length of Sides $L_n = \frac{1}{3} L_{n-1}$</th>
<th>Perimeter of $n^{th}$ Iterate $P_n = N_n \cdot L_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0 = 3$</td>
<td>$L_0 = S$</td>
<td>$P_0 = 3 \cdot S$</td>
</tr>
<tr>
<td>$N_1 = 4 \cdot 3$</td>
<td>$L_1 = \frac{1}{3} S$</td>
<td>$P_1 = 4 \cdot S$</td>
</tr>
<tr>
<td>$N_2 = 4^2 \cdot 3$</td>
<td>$L_2 = \frac{1}{3^2} S$</td>
<td>$P_2 = \frac{4^2}{3} \cdot S$</td>
</tr>
<tr>
<td>$N_3 = 4^3 \cdot 3$</td>
<td>$L_3 = \frac{1}{3^3} S$</td>
<td>$P_3 = \frac{4^3}{3^2} \cdot S$</td>
</tr>
<tr>
<td>$N_4 = 4^4 \cdot 3$</td>
<td>$L_4 = \frac{1}{3^4} S$</td>
<td>$P_4 = \frac{4^4}{3^3} \cdot S$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$N_n = 4^n \cdot 3$</td>
<td>$L_n = \frac{1}{3^n} S$</td>
<td>$P_n = \frac{4^n}{3^{n+1}} \cdot S$</td>
</tr>
</tbody>
</table>

Observe that $P_n = 4 \left(\frac{4}{3}\right)^{n-1} \cdot S \to +\infty$ as $n \to \infty$

Therefore, the snowflake has infinite length.
Area of n\textsuperscript{th} iterate:

\[ A_n = A_{n-1} + N_{n-1} \times \left( \frac{1}{9} \right) A_0 \]

For \( n = 1, 2, 3, \ldots \):

\[ A_0 = \frac{\sqrt{3}}{4} \leq 2 \]

\[ A_1 = A_0 + N_0 \cdot \left( \frac{1}{9} \right) A_0 = A_0 + 3 \left( \frac{1}{9} \right) A_0 = \frac{4}{3} A_0 \]

\[ A_2 = A_1 + N_1 \cdot \left( \frac{1}{9} \right)^2 A_0 = \frac{4}{3} A_0 + 4 \cdot 3 \left( \frac{1}{9} \right)^2 A_0 = \frac{4}{3} A_0 + \frac{1}{3} \left( \frac{4}{9} \right) A_0 \]

\[ A_3 = A_2 + N_2 \cdot \left( \frac{1}{9} \right)^3 A_0 = \left[ \frac{4}{3} A_0 + \frac{1}{3} \left( \frac{4}{9} \right) A_0 \right] + \left( 4 \cdot 3 \cdot 3 \right) \left( \frac{1}{9} \right)^3 A_0 \]

\[ = \frac{4}{3} A_0 + \frac{1}{3} \left[ \left( \frac{4}{9} \right) A_0 + \left( \frac{1}{9} \right)^2 A_0 \right] \]

\[ A_4 = A_3 + N_3 \cdot \left( \frac{1}{9} \right)^4 A_0 \]

\[ = \frac{4}{3} A_0 + \frac{1}{3} \left[ \left( \frac{4}{9} \right) A_0 + \left( \frac{1}{9} \right)^2 A_0 + \left( \frac{1}{9} \right)^3 A_0 \right] \]

\[ \vdots \]

\[ A_n = \frac{4}{3} A_0 + \frac{1}{3} \sum_{k=1}^{n-1} \left( \frac{4}{9} \right)^k A_0 \]

Geometric Sum converging to:

\[ \frac{\frac{4}{3} A_0}{1 - \frac{4}{9}} = \frac{4}{5} A_0 \text{ as } n \to +\infty. \]

Thus,

\[ A_n \to \frac{4}{3} A_0 + \frac{1}{3} \left( \frac{4}{5} A_0 \right) = \frac{24}{15} A_0 = \frac{8}{5} A_0 \text{ as } n \to +\infty. \]
Some beautiful tilings, a few examples of which are illustrated above, can be made with iterations toward Koch snowflakes.

In addition, two sizes of Koch snowflakes in area ratio 1:3 tile the plane region shown above.
The fractal image known as the Sierpiński triangle, shown here, may be constructed as the limit of a simple continuous curve in the plane. It can be formed by a process of repeated modification in a manner analogous to that used in the construction of the Koch snowflake:

**Construction**

Take the initial curve in the construction to be a line segment that forms the base of an equilateral triangle in the plane.

At each recursive stage, replace each line segment on the curve with three shorter ones, each of equal length, such that:

1. the three line segments replacing a single segment from the previous stage always make 120° angles at each junction between two consecutive segments, with the first and last segments of the curve either parallel to the base of the given equilateral triangle or forming a 60° angle with it.

2. no pair of line segments forming the curve at any stage ever intersect, except possibly at their endpoints.

3. every line segment of the curve remains on, or within, the initial equilateral triangle but outside the central downward pointing equilateral triangular regions that are external to the limiting curve displayed above.

The resulting fractal curve is called the Sierpiński arrowhead curve, and its limiting shape is the Sierpiński triangle.

Evidently, in the case of the Sierpiński triangle, the scaling factor is \( r = \frac{1}{2} \) and \( N = 3 \).

I.e., when any line segment of the Sierpiński arrowhead curve is scaled down, it is replaced by \( N = 3 \) line segments, each being \( r = \) one-half the length of the previous one. So the fractal dimension for this curve must be \( D \) where:

\[
\left( \frac{1}{r} \right)^D = N \iff 2^D = 3 \iff D \cdot \ln 2 = \ln 3 \iff D = \frac{\ln 3}{\ln 2} \approx 1.585
\]
Scaling Ratios and Fractal Dimension

Suppose we begin with a line segment and divide it into shorter pieces of equal length.

<table>
<thead>
<tr>
<th>Scaling Ratio</th>
<th># Resulting Pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>2</td>
</tr>
<tr>
<td>1/3</td>
<td>3</td>
</tr>
<tr>
<td>1/4</td>
<td>4</td>
</tr>
</tbody>
</table>

Note that \( \left( \frac{1}{r} \right)^1 = N \) for any scaling ratio \( r \) for the line segment.

We say that the segment is 1-dimensional as it possesses only length.

Now divide a square into sub-squares, each having equal area.

<table>
<thead>
<tr>
<th>Scaling Ratio</th>
<th># Resulting &quot;pieces&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>4</td>
</tr>
<tr>
<td>1/3</td>
<td>9</td>
</tr>
<tr>
<td>1/4</td>
<td>16</td>
</tr>
</tbody>
</table>

Note that \( \left( \frac{1}{r} \right)^2 = N \) for any scaling ratio \( r \) for the square.

We say that the square is 2-dimensional possessing both length and width.

Upon dividing a cube into sub-cubes of equal volume, we say:

<table>
<thead>
<tr>
<th>Scaling Ratio</th>
<th># Resulting &quot;pieces&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>8</td>
</tr>
<tr>
<td>1/3</td>
<td>27</td>
</tr>
<tr>
<td>1/4</td>
<td>64</td>
</tr>
</tbody>
</table>
Note that \((\frac{1}{r})^3 = N\) for any scaling ratio \(r\) in the case of the cube, which makes the cube 3-dimensional as it has length, width, and height.

<table>
<thead>
<tr>
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<td>(\frac{1}{2})</td>
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<tr>
<td>(\frac{1}{4})</td>
<td>64</td>
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</tbody>
</table>

These examples help motivate the definition of fractal dimension. Recall that a fractal curve is defined iteratively, such a figure cannot be drawn precisely with a pencil, as it is infinitely complex. In order to compare the complexities of fractal curves, we calculate their fractal dimensions.

Say that a fractal curve has dimension \(D\) if

\[ (\frac{1}{r})^D = N \]

where \(r\) = scaling ratios and

\(N = \# \text{ pieces that result from scaling} \)

Consider the Koch curve

\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \quad 1 \]

First, \((\frac{1}{r})^D = N\)

\[ (\frac{1}{(\frac{1}{3})})^D = 4 \]

Thus, \(3^D = 4\)

\[ D = \frac{\ln 4}{\ln 3} = 1.262 \]
The first 6 iterations in the Hilbert curve construction are displayed below. The limiting Hilbert curve will pass through every point of the square. I.e., it is an example of space-filling curve and was devised by the mathematician David Hilbert. The Hilbert curve has fractal dimension 2.
An Escheresque fractal by Peter Raedschelders

A fractal spiral created by Turtle Graphics

*Visage of War* (1940) by Salvador Dalí