A fractal is a mathematical set that exhibits a repeating pattern displayed at every scale. It is also known as expanding symmetry or evolving symmetry. If the replication is exactly the same at every scale, it is called a self-similar pattern. A 3-dimensional example of such self-similarity occurs in the Menger Sponge.

Fractals can also be nearly the same at different levels. This latter pattern is illustrated in small magnifications of the Mandelbrot set. Fractals are different from other geometric figures because of the way in which they scale. As mathematical equations, plane fractals are usually nowhere differentiable. The term fractal was first used by mathematician Benoît Mandelbrot in 1975. He based it on the Latin frāctus meaning broken or fractured. Fractal patterns with various degrees of self-similarity can be found in nature, technology and art.

Koch Snowflake

The Koch snowflake (also known as the Koch star and Koch island) is a mathematical curve and one of the earliest fractals to have been described. The Koch snowflake is based on the Koch curve, which appeared in a 1904 paper titled "On a continuous curve without tangents, constructible from elementary geometry" by the Swedish mathematician Helge von Koch.

Construction

The Koch curve can be constructed by starting with an equilateral triangle, then recursively altering each line segment that forms a side of the figure as follows:

1. divide the line segment into three segments of equal length.
2. draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
3. remove the line segment that is the base of the triangle from step 2.
4. After one iteration of this process, the result is a shape similar to the Star of David.
5. The Koch curve is the limit approached as the above steps are followed over and over again.
Properties

1. The Koch curve has **infinite length** because each time the above steps are applied to each line segment of the figure then the resulting figure will possess 4 times as many line segments as before. The length of each new segment being \((1/3)^n\) the length of each segment from the previous stage. Hence, the total length of the figure increases by \((4/3)^n\) at each iteration. Thus, the length of the figure at the \(n^{th}\) iteration will be \((4/3)^n\) times the perimeter of the original triangle.

As \(n \to +\infty\) then \((4/3)^n \to +\infty\) as well, so the limiting curve has infinite perimeter.

2. The Koch curve is continuous everywhere but differentiable nowhere.

3. As noted above, when any line segment of the Koch curve is scaled down, it is replaced by: 

\[ N = 4 \text{ segments, each being } r = (1/3)^n \text{ the length of the previous one.} \]

We call \(r\) the **scaling factor** for this curve, and by definition, the **fractal dimension** of the Koch curve is \(D\) where:

\[
\left( \frac{1}{r} \right)^D = N \iff 3^D = 4 \iff D \cdot \ln 3 = \ln 4 \iff D = \frac{\ln 4}{\ln 3} \approx 1.262.
\]

(\textit{the higher the fractal dimension - the greater the degree of complexity of the curve})

4. Taking \(s\) as the side length of the initial equilateral triangle, then the original triangle’s area is: 

\(A_0 = \frac{s^2 \sqrt{3}}{4}\). At every successive iterate, recall that the side length of each newly constructed triangle is \((1/3)^n\) the side length from the previous iteration. Since the area of each triangle constructed at the \(n^{th}\) step is proportional to the square of its side length, then the area of each triangle constructed at that step must be \((1/9)^n\) the area of each triangle constructed at the \((n-1)^{th}\) step. In each iteration after the first, 4 times as many triangles are constructed than in the previous iteration, and because the first iteration adds 3 triangles, then the \(n^{th}\) iteration must add a total of \(3 \cdot 4^{(n-1)}\) triangles. Combining these two formulas yields the recursive area formula:

\[A_n = A_{n-1} + \frac{3 \cdot 4^{n-1}}{9^n} A_0, \ n \geq 1, \text{ where } A_0 \text{ is area of the original triangle.}\]

Substituting in \(A_1 = \frac{4}{3} A_0\), and expanding (see details on page 5) yields:

\[A_n = \frac{4}{3} A_0 + \sum_{k=2}^{n} \left( \frac{3 \cdot 4^{k-1}}{9^k} A_0 \right) = \left( \frac{4}{3} + \frac{1}{3} \sum_{k=2}^{n} \frac{3 \cdot 4^{k-1}}{9^k} \right) \cdot A_0 \]

\[= \left( \frac{4}{3} + \frac{1}{3} \sum_{k=2}^{n} \frac{9 \cdot 4^{k-1}}{9^k} \right) \cdot A_0 = \left( \frac{4}{3} + \frac{1}{3} \sum_{k=2}^{n} \frac{4^{k-1}}{9^{k-1}} \right) \cdot A_0 = \left( \frac{4}{3} + \frac{1}{3} \sum_{k=1}^{n-1} \frac{4^k}{9^k} \right) \cdot A_0 \]

In the limit, as \(n\) goes to infinity, the sum of the powers of \(4/9\) in the formula above converges to \(4/5\).

Hence, \(\lim_{n \to +\infty} A_n = \left( \frac{4}{3} + \frac{1}{3} \cdot \frac{4}{5} \right) \cdot A_0 = \frac{8}{5} A_0\)

So the area of a Koch snowflake is \(8/5\) times the area of the original triangle, or \(\frac{2s^2 \sqrt{3}}{5}\) sq units.
Therefore, the Koch fractal snowflake has an **infinite perimeter** yet it encloses a **finite area**.

Some beautiful tilings, a few examples of which are illustrated below, can be made with iterations toward Koch snowflakes.

In addition, two sizes of Koch snowflakes in an area ratio of 1:3 tile the plane region shown below.
Specifications of the $n^{th}$ iterate of the snowflake curve for $n = 0, 1, 2, \ldots$

<table>
<thead>
<tr>
<th>Number of sides $N_n = 4 \cdot N_{n-1}$</th>
<th>Length of each side $L_n = \frac{1}{3^n} \cdot L_{n-1}$</th>
<th>Perimeter of $n^{th}$ iterate $P_n = N_n \cdot L_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0 = 3$</td>
<td>$L_0 = s$</td>
<td>$P_0 = 3 \cdot s$</td>
</tr>
<tr>
<td>$N_1 = 4 \cdot 3$</td>
<td>$L_1 = \frac{1}{3} s$</td>
<td>$P_1 = 4 \cdot s$</td>
</tr>
<tr>
<td>$N_2 = 4^2 \cdot 3$</td>
<td>$L_2 = \frac{1}{3^2} s$</td>
<td>$P_2 = \frac{4^2}{3} \cdot s$</td>
</tr>
<tr>
<td>$N_3 = 4^3 \cdot 3$</td>
<td>$L_3 = \frac{1}{3^3} s$</td>
<td>$P_3 = \frac{4^3}{3^2} \cdot s$</td>
</tr>
<tr>
<td>$N_4 = 4^4 \cdot 3$</td>
<td>$L_4 = \frac{1}{3^4} s$</td>
<td>$P_4 = \frac{4^4}{3^3} \cdot s$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$N_n = 4^n \cdot 3$</td>
<td>$L_n = \frac{1}{3^n} s$</td>
<td>$P_n = \frac{4^n}{3^{n-1}} \cdot s$</td>
</tr>
</tbody>
</table>

Observe that the perimeter: $P_n = 4 \left( \frac{4}{3} \right)^{n-1} \cdot s \to +\infty$ as $n \to +\infty$.

Therefore, the boundary of the Koch fractal snowflake curve has infinite length.
Area enclosed by the \( n \)th iterate of the snowflake curve:

\[
A_n = A_{(n-1)} + N_{(n-1)} \times \left( \frac{1}{9} \right)^n \cdot A_0 \quad \text{for } n = 1, 2, 3, \ldots
\]

\[
A_0 = \frac{\sqrt{3}}{4} s^2
\]

\[
A_1 = A_0 + N_0 \cdot \left( \frac{1}{9} \right) A_0
\]

\[
A_1 = A_0 + 3 \cdot \left( \frac{1}{9} \right) A_0 = \frac{4}{3} A_0
\]

\[
A_2 = A_1 + N_1 \cdot \left( \frac{1}{9} \right)^2 A_0
\]

\[
A_2 = \frac{4}{3} A_0 + 4 \cdot 3 \cdot \left( \frac{1}{9} \right)^2 A_0 = \frac{4}{3} A_0 + \frac{1}{3} \cdot \left( \frac{4}{9} \right) A_0
\]

\[
A_3 = A_2 + N_2 \cdot \left( \frac{1}{9} \right)^3 A_0
\]

\[
A_3 = \frac{4}{3} A_0 + \frac{1}{3} \left[ \left( \frac{4}{9} \right) A_0 + \left( \frac{4}{9} \right)^2 A_0 \right]
\]

\[
A_4 = A_3 + N_3 \cdot \left( \frac{1}{9} \right)^4 A_0
\]

\[
A_4 = \frac{4}{3} A_0 + \frac{1}{3} \left[ \left( \frac{4}{9} \right) A_0 + \left( \frac{4}{9} \right)^2 A_0 + \left( \frac{4}{9} \right)^3 A_0 \right]
\]

\[
\begin{align*}
A_n &= \frac{4}{3} A_0 + \frac{1}{3} \left[ \sum_{k=1}^{n} \left( \frac{4}{9} \right)^k A_0 \right] \\
&= \frac{4}{3} A_0 + \frac{1}{3} \cdot \frac{\left( \frac{4}{9} \right)^{n+1} - \frac{4}{9}}{1 - \frac{4}{9}} A_0 \\
&= \frac{4}{3} A_0 + \frac{8}{5} A_0 = \frac{4}{5} A_0
\end{align*}
\]

Thus, the limiting area of the Koch snowflake is given by the value:

\[
A_n \rightarrow \frac{4}{3} A_0 + \frac{1}{3} \cdot \left( \frac{4}{9} \right) A_0 = \frac{24}{15} A_0 = \frac{8}{5} A_0 \quad \text{as } n \rightarrow +\infty.
\]
The Sierpiński Triangle or Sierpiński Gasket

The fractal image known as the Sierpiński triangle, shown here, may be constructed as the limit of a simple continuous curve in the plane. It can be formed by a process of repeated modification in a manner analogous to that used in the construction of the Koch snowflake:

Construction
Take the initial curve in the construction to be a line segment that forms the base of an equilateral triangle in the plane.

At each recursive stage, replace each line segment on the curve with three shorter ones, each of equal length, such that:
1. the three line segments replacing a single segment from the previous stage always make 120° angles at each junction between two consecutive segments, with the first and last segments of the curve either parallel to the base of the given equilateral triangle or forming a 60° angle with it.
2. no pair of line segments forming the curve at any stage ever intersect, except possibly at their endpoints.
3. every line segment of the curve remains on, or within, the given equilateral triangle but outside the central downward pointing equilateral triangular regions that are external to the limiting curve displayed above.

The resulting fractal curve is called the Sierpiński arrowhead curve, and its limiting shape is the Sierpiński triangle.

Evidently, in the case of the Sierpiński triangle, the scaling factor is \( r = \frac{1}{2} \) and \( N = 3 \).

I.e., when any line segment of the Sierpiński arrowhead curve is scaled down, it is replaced by \( N = 3 \) line segments, each being \( r = \frac{1}{2} \) the length of the previous one. So the fractal dimension for this curve must be \( D \) where:

\[
\left( \frac{1}{r} \right)^{D} = N \iff 2^{D} = 3 \iff D \cdot \ln 2 = \ln 3 \iff D = \frac{\ln 3}{\ln 2} \approx 1.585
\]
Details on Scaling Ratios and Fractal Dimensions

Suppose we begin with a line segment and divide it successively into shorter and shorter pieces of equal length.

Note that \( \left( \frac{1}{r} \right)^1 = N \) for any scaling ratio \( r \) for the line segment.

Here, \( N \) denotes the number of pieces, or shorter line segments, that result from the scaling.

We say that the segment is 1–dimensional as it possesses only length.

We now successively partition a square into smaller sub-squares, each of equal area.

Note that \( \left( \frac{1}{r} \right)^2 = N \) for any scaling ratio \( r \) of the square.

Here, \( N \) denotes the number of pieces, or sub-squares, that result from the scaling.

We say that the square is 2–dimensional as it possesses both length and width.

Similarly, we may proceed to divide a cube successively into sub-cubes of equal volume.
Observe that in the case of the cube, it follows that

\[(\frac{1}{r})^3 = N\]

for any scaling ratio \(r\), where \(N\) denotes the number of pieces, or sub-cubes, that results from the scaling.

We say that the cube is 3-dimensional as it has length, width and height.

These examples help to motivate the definition of \textbf{fractal dimension}. Recall that a fractal curve is defined iteratively, such a curve cannot be drawn precisely with a pencil as it is infinitely complex. In order to compare the complexities of fractal curves, we calculate their fractal dimensions.

<table>
<thead>
<tr>
<th>Scaling ratio (r)</th>
<th># resulting pieces (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{2})</td>
<td>8</td>
</tr>
<tr>
<td>(\frac{1}{3})</td>
<td>27</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>64</td>
</tr>
</tbody>
</table>

We say that a fractal curve has \textbf{dimension} \(D \iff \left(\frac{1}{r}\right)^D = N\)

where \(r = \) scaling ratio and \(N = \# \) pieces that result from scaling.

Now recall the iterative construction of the Koch snowflake curve:

Note that at every stage, \(r = \frac{1}{3}\) and \(N = 4\).

Thus,

\[\left(\frac{1}{r}\right)^D = N \iff \left(\frac{1}{1/3}\right)^D = 4\]

\[3^D = 4\]

\[\iff \ln (3^D) = \ln (4)\]

\[D \cdot \ln (3) = \ln (4)\]

\[\therefore D = \frac{\ln (4)}{\ln (3)} \approx 1.262\] is its fractal dimension.
By comparison, in the iterative construction of the Sierpinski Triangle, we find that

\[ r = \frac{1}{2} \quad \text{and} \quad N = 3 \text{ at each stage.} \]

Thus,

\[ \left( \frac{1}{r} \right)^D = N \iff \left( \frac{1}{\frac{1}{2}} \right)^D = 3 \]

\[ 2^D = 3 \]

\[ \iff \ln \left( 2^D \right) = \ln \left( 3 \right) \]

\[ D \cdot \ln \left( 2 \right) = \ln \left( 3 \right) \]

\[ \therefore D = \frac{\ln \left( 3 \right)}{\ln \left( 2 \right)} \approx 1.585 \text{ is its fractal dimension.} \]

The Hilbert Curve

The first 4 iterations in the Hilbert curve construction are displayed below. The limiting Hilbert curve will pass through every point of the square. I.e., it is an example of space-filling curve and was devised by the mathematician David Hilbert.

Observe that, at any stage, when the curve is scaled down by a factor of \( r = \frac{1}{2} \) then the number of pieces, or copies of the prior iteration of the curve which result, is always \( N = 4 \).

That is, in general, at the \( k^{th} \) stage of the construction

\[ r = \left( \frac{1}{2} \right)^k \quad \text{and} \quad N = 4^k = 2^{2k} \quad \text{for any} \quad k = 1, 2, 3, \ldots \]

Thus,

\[ \left( \frac{1}{r} \right)^D = N \iff \left( \frac{1}{\left( \frac{1}{2} \right)^k} \right)^D = 2^{2k} \]

\[ 2^{kD} = 2^{2k} \]

\[ \iff k \cdot D = 2k \]

\[ \therefore D = \frac{2k}{k} = 2 \text{ is the fractal dimension of the Hilbert Curve.} \]
An Escheresque fractal by Peter Raedschelders

A fractal spiral created by Turtle Graphics

*Visage of War*  
(1940) by Salvador Dalí