IMPROPER INTEGRALS

Problem 1: Construct a continuous function $f$ which is such that

$$\int_0^\infty f(x) \, dx \text{ converges but } \lim_{x \to \infty} f(x) \text{ does not exist .}$$

Suppose that the graph of $f$ is piecewise linear with corners at the points

$$P_i = (s_i, f(s_i)) \text{ for } i = 0, 1, 2, 3, \ldots$$

where $s_0 = 0$, $f(0) = -1$, and $s_i = \sum_{k=1}^{i} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i}$, for any $i = 1, 2, 3, \ldots$.

and $f(s_{2i}) = -1$, $f(s_{2i-1}) = +1$ for any $i = 1, 2, 3, \ldots$

( It will be shown in Chapter 12 that $s_i$ increases without bound as $i \to \infty$.)

We display a graph of $y = f(x)$ below.

If we integrate the function $f$ over the interval $[0, b]$, where $b$ is any positive real number, then it is geometrically evident that
\[ \int_{0}^{b} f(x) \, dx \] will be \( \leq 0 \) \( \iff \) the line \( x = b \) intersects a green shaded region in the graph below.

and \[ \int_{0}^{b} f(x) \, dx \] will be \( \geq 0 \) \( \iff \) the line \( x = b \) intersects a blue shaded region in the graph below.

Moreover, for any real number \( b > 1 \):

observe that if \( b \in [s_n, s_{n+1}) \) for some odd \( n \) then the line \( x = b \) will intersect a blue region and

\[
0 \leq \int_{0}^{b} f(x) \, dx \leq \frac{1}{2} \left[ \frac{s_{n+1} - s_n}{2} \right] = \frac{1}{4} \frac{1}{n+1}.
\]

Otherwise, \( b \in [s_n, s_{n+1}) \) for some even \( n \) so the line \( x = b \) intersects a green region and

\[
0 \geq \int_{0}^{b} f(x) \, dx \geq -\frac{1}{2} \left[ \frac{s_{n+1} - s_n}{2} \right] = -\frac{1}{4} \frac{1}{n+1}.
\]

In any case,

\[ b \in [s_n, s_{n+1}) \Rightarrow 0 \leq \left| \int_{0}^{b} f(x) \, dx \right| \leq \frac{1}{4} \frac{1}{n+1}.
\]

Therefore,

\[ \lim_{b \to \infty} \left| \int_{0}^{b} f(x) \, dx \right| = 0 \Rightarrow \int_{0}^{\infty} f(x) \, dx = 0 \]

even though \( \lim_{x \to \infty} f(x) \) does not exist.
Problem 2: Find a function \( f \) which is such that 
\[
\int_0^\infty f(x) \, dx \text{ converges but } f \text{ is unbounded on } [0, \infty).
\]

The Fresnel integrals

\[
(1) \quad \int_0^\infty \cos(x^2) \, dx \quad \text{and} \quad (2) \quad \int_0^\infty \sin(x^2) \, dx
\]

occur frequently in the theory of optics (the diffraction of light waves). In advanced calculus
it is shown that both integrals converge to the number \( \frac{\sqrt{\pi}}{2 \sqrt{2}} \).

Now consider the integral \( I = \int_0^\infty x \cos(x^4) \, dx \). It is related to (1).

Note that the function \( C(x) = x \cos(x^4) \) is unbounded on \([0, +\infty)\) and has the following graph:

Upon setting \( t = x^2 \) in the integral \( I \) we have that \( dt = 2x \, dx \) so

\[
I = \int_0^\infty x \cos(x^4) \, dx = \int_0^\infty \left[ \frac{1}{2} \right] \cos(t^2) \, dt = \frac{1}{2} \frac{\sqrt{\pi}}{2 \sqrt{2}} = \frac{\sqrt{\pi}}{4 \sqrt{2}}
\]

even though \( |C((n \pi)^{1/4})| \) increases without bound as \( n \to \infty \) through integer values.
The preceding problems illustrate the fact that
\[ \int_a^\infty f(x) \, dx \text{ convergent} \quad \text{does not imply} \quad \lim_{x \to \infty} f(x) \text{ exists.} \]

However, we do have the following result.

**Lemma:** If the function \( f \) is continuous on \( [a, \infty) \),
\[ \lim_{x \to \infty} f(x) \text{ exists and } \int_a^\infty f(x) \, dx \text{ converges} \]
Then
\[ \lim_{x \to \infty} f(x) = 0. \]

This result is easy to verify from a picture, since if \( \lim_{x \to \infty} f(x) = L \not= 0 \), then for any \( \varepsilon > 0 \) there must correspond a number \( N > a \) such that \( 0 < \left| f(x) - L \right| < \varepsilon \) for all \( x \geq N \).

Note that in the figure, \( \lim_{b \to \infty} \int_N^b f(x) \, dx \geq \lim_{b \to \infty} \int_N^b M \, dx = +\infty \).

Hence,
\[ \lim_{b \to \infty} \int_N^b f(x) \, dx \text{ diverges which implies that } \int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx \text{ diverges also.} \]