THE CATENARY

A derivation of the equation of a hanging cable. (*Catenary* is derived from the Latin word for *chain*).

Consider the section AP from the lowest point A to a general point $P = (x, y)$ on the cable (see the figure below) and imagine the rest of the cable to have been removed. The forces acting on the cable are:

- **i.** H = horizontal tension pulling at A
- **ii.** T = tangetial tension pulling at P
- **iii.** $W = \delta s$ = weight of s feet of cable of density δ pounds per foot.

To be in equilibrium, the horizontal and vertical components of T must just balance H and W respectively. Thus,

$$T \cos \phi = H \quad \text{and} \quad T \sin \phi = W = \delta s .$$

So

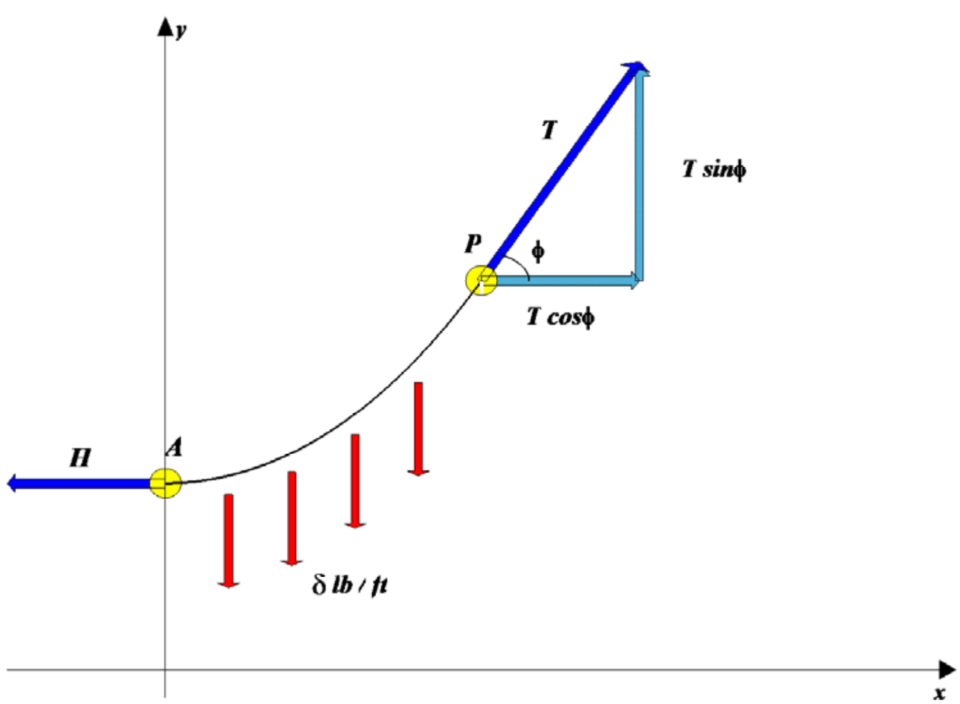
$$\frac{T \sin \phi}{T \cos \phi} = \tan \phi = \frac{\delta s}{H} .$$

But since $\tan \phi = \dfrac{dy}{dx}$, we see that

$$\frac{dy}{dx} = \frac{\delta s}{H} .$$

Therefore, upon differentiating both sides with respect to x, we obtain the second order DE

$$(*) \qquad \frac{d^2 y}{d x^2} = \frac{\delta}{H}\frac{ds}{dx} = \frac{\delta}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} .$$



Problem ____

Show that $\boldsymbol{y = a \cosh\left(\frac{x}{a}\right) + C}$ satisfies the differential equation (*) with $a = \frac{H}{\delta}$ by doing the following:

1. Substitute $z = \frac{dy}{dx}$ and $\frac{dz}{dx} = \frac{d^2 y}{d x^2}$ in (*) and obtain a first order DE with dependent variable z .

2. Solve the first order DE in z .

Hint: This involves verifying that: $\int \frac{1}{\sqrt{1+z^2}}\, dz = \ln(z + \sqrt{1+z^z}) + C$.

Then note that

$\ln(z + \sqrt{1+z^z}) = \sinh^{-1}(z)$, the **inverse hyperbolic sine** of z .

3. Replace z in terms of y and solve the resulting first order DE for y .