Absolute Convergence vs. Conditional Convergence

The alternating harmonic series:

\[ \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \]

converges by the Alternating Series Test to a number, \( S \), where

\[ \frac{1}{1} - \frac{1}{2} \leq S \leq \frac{1}{1} - \frac{1}{2} + \frac{1}{3} \Rightarrow \frac{1}{2} \leq S \leq \frac{5}{6}. \]

When studying power series, we will show that \( S = \ln 2 \). That is,

\[ (1) \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2 \]

Multiply (1) term-by-term by \( \frac{1}{2} \) to obtain the series:

\[ (2) \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \frac{1}{18} - \frac{1}{20} + \frac{1}{22} - \frac{1}{24} + \ldots = \frac{1}{2} \ln 2 \]

Now insert zeros between successive terms in series (2) to obtain the series:

\[ (3) \quad \sum_{n=1}^{+\infty} \frac{(-1) \cdot \cos \left( \frac{n\pi}{2} \right)}{n} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + 0 - \frac{1}{12} + \ldots = \frac{1}{2} \ln 2 \]

Adding: (1) + (3) term-by-term, we find:

\[ (4) \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} - \cos \left( \frac{n\pi}{2} \right)}{n} = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \frac{1}{11} - \frac{1}{6} + \ldots \]

\[ = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{+\infty} \frac{(-1) \cdot \cos \left( \frac{n\pi}{2} \right)}{n} = \ln 2 + \frac{1}{2} \ln 2 = \frac{3}{2} \ln 2 \]

Remark: Series (4) contains the same nonzero terms as does series (1), just rearranged, so that one negative term occurs after each pair of successive positive terms. However, the sums of these series are not the same.

Thus, “infinite addition” is not generally commutative in case the terms in the series are not all of the same sign.
Some Theorems on Absolute and Conditional Convergence of Infinite Series

Definitions: (i) If \[ \sum_{n=1}^{\infty} |a_n| \] converges then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

(ii) If \( \sum_{n=1}^{\infty} a_n \) converges but \( \sum_{n=1}^{\infty} |a_n| \) diverges

then \( \sum_{n=1}^{\infty} a_n \) is conditionally convergent.

Theorem 1: If \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent then \( \sum_{n=1}^{\infty} a_n \) is convergent.

Proof: Since \( \sum_{n=1}^{\infty} |a_n| \) converges then \( \sum_{n=1}^{\infty} 2|a_n| \) converges by the Constant Multiple Rule for convergent series.

It is also true that \( 0 \leq a_n + |a_n| \leq 2|a_n| \) for all \( n = 1, 2, 3, \ldots \).

So the series \( \sum_{n=1}^{\infty} (a_n + |a_n|) \) converges by the Direct Comparison Test.

Consequently, the series \( \sum_{n=1}^{\infty} [(a_n + |a_n|) - |a_n|] = \sum_{n=1}^{\infty} a_n \) also converges by the Term-by-Term Subtraction Property for convergent series.

Theorem 2: Positive and Negative Parts of Absolutely and Conditionally Convergent Series

Let \( \{a_n\}_{n=1}^{\infty} \) be any sequence of real numbers.

Set \( a_n^+ = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} = \frac{a_n + |a_n|}{2} \)

and \( a_n^- = \begin{cases} 0 & \text{if } a_n \geq 0 \\ a_n & \text{if } a_n < 0 \end{cases} = \frac{a_n - |a_n|}{2} \) for all \( n = 1, 2, 3, \ldots \).

(i) If \( \sum_{n=1}^{\infty} a_n \) converges absolutely then both \( \sum_{n=1}^{\infty} (a_n^+) \) and \( \sum_{n=1}^{\infty} (a_n^-) \) converge.

(ii) If \( \sum_{n=1}^{\infty} a_n \) converges conditionally then both \( \sum_{n=1}^{\infty} (a_n^+) \) and \( \sum_{n=1}^{\infty} (a_n^-) \) diverge.
Proof

(i) In this case, \( \sum_{n=1}^{\infty} |a_n| \) converges, and since

\[
0 \leq \frac{a_n + |a_n|}{2} = a_n^+ \leq |a_n|
\]

and

\[
0 \leq \frac{|a_n| - a_n}{2} = -a_n^- \leq |a_n| \quad \text{for all} \quad n = 1, 2, 3, \ldots
\]

then both \( \sum_{n=1}^{\infty} (a_n^+) \) and \( \sum_{n=1}^{\infty} (a_n^-) \) must converge by Direct Comparison with \( \sum_{n=1}^{\infty} |a_n| \).

Therefore, both \( \sum_{n=1}^{\infty} (a_n^+) \) and \( \sum_{n=1}^{\infty} (a_n^-) \) are convergent whenever \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

(ii) In this case, \( \sum_{n=1}^{\infty} a_n \) converges and yet \( \sum_{n=1}^{\infty} |a_n| \) diverges.

So both \( \sum_{n=1}^{\infty} (a_n^+ + |a_n|) \) and \( \sum_{n=1}^{\infty} (a_n^- - |a_n|) \) are divergent by the Term-by-Term Addition and Subtraction Theorem.

Therefore, \( \sum_{n=1}^{\infty} 2(a_n^+) \) and \( \sum_{n=1}^{\infty} 2(a_n^-) \) diverge \( \Rightarrow \sum_{n=1}^{\infty} (a_n^+) \) and \( \sum_{n=1}^{\infty} (a_n^-) \) diverge, since term-by-term multiplication by a nonzero constant has no effect on the convergence of an infinite series.

Theorem 3: On Rearranging the Terms of an Infinite Series

(i) If \( \sum_{n=1}^{\infty} a_n \) converges absolutely and we arbitrarily rearrange its terms to obtain the series \( \sum_{n=1}^{\infty} b_n \) then \( \sum_{n=1}^{\infty} b_n \) is also convergent and \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n \).

(ii) If \( \sum_{n=1}^{\infty} a_n \) is only conditionally convergent, then we may rearrange its terms to obtain a new series which

- converges to any specified real number,
- diverges to \(+\infty\)
- diverges to \(-\infty\).