THEOREM 1:  \( \sqrt{2} \) is an irrational number.

Proof:

In order to prove this theorem, we begin by assuming to the contrary that \( \sqrt{2} \) is a rational number. Then \( \sqrt{2} = \frac{p}{q} \) for some positive integers \( p \) and \( q \).

From the Pythagorean Theorem,

\[
\left( \frac{p}{q} \right)^2 = 1^2 + 1^2 = 2.
\]

So

\[
p^2 = 2q^2 \quad (1)
\]

Now we factor out as many 2’s as possible from \( p \) and then from \( q \) to obtain:

\[
p = 2^a p_1 \quad \text{and} \quad q = 2^b q_1
\]

where \( p_1 \) and \( q_1 \) are necessarily odd (why?).

Substituting these expressions for \( p \) and \( q \) into (1) we find that

\[
2^{2a} \cdot p_1^2 = 2 \cdot (2^{2b} q_1^2)
\]

\[\Rightarrow \quad 2^{2a} \cdot p_1^2 = 2^{(2b+1)} q_1^2 \quad (2)\]

Note that \( 2a \neq 2b+1 \) since \( 2a \) is even but \( 2b+1 \) is odd.

So there are two cases to consider:

Case 1: \( 2a > 2b+1 \).

Then (2) \( \Rightarrow \quad 2^{(2a-2b-1)} \cdot p_1^2 = q_1^2 \).

Therefore, \( q_1^2 \) must be even.

But since \( q_1 \) is odd, \( q_1 = 2k + 1 \) for some integer \( k \)

\[
\text{and} \quad q_1^2 = 4k^2 + 4k + 1 = 2[2(k^2 + k)] + 1 \]

i.e. \( q_1^2 = 2m + 1 \) for some positive integer \( m \).

\[\Rightarrow q_1^2 \text{ is odd}.\]
Thus, \( q_1^2 \) is both even and odd. – This is impossible. Consequently, \( 2a \) cannot be larger than \( 2b+1 \).

Case 2: Suppose \( 2a < 2b+1 \).

From an argument analogous to that given in Case 1 (which is left for the reader to formulate) we may conclude that \( p_1^2 \) is both even and odd. Of course, this is not possible. Consequently, \( 2a \) cannot be smaller than \( 2b+1 \).

Therefore, from our results in cases 1 and 2 above, we must conclude that our initial assumption:
\[
\sqrt{2} = \frac{p}{q}
\]
for some positive integers \( p \) and \( q \)

was false.

Hence, \( \sqrt{2} \) is an irrational number.

THEOREM 2: The square root of any natural number that is not a perfect square is irrational.

The following proof was originally proposed by the famous German mathematician, Richard Dedekind (October 6, 1831 – February 12, 1916).

Assume \( D \) is a non-square natural number then there is an integer \( n \) such that:
\[
n^2 < D < (n+1)^2 \quad (1)
\]
Assume also that \( \sqrt{D} \) is a rational number:
\[
\sqrt{D} = \frac{p}{q}
\]
where \emph{q is the smallest natural number for which this is true*}, then
\[
Dq^2 = p^2 \quad (2)
\]
Multiplying (1) through by \( q^2 \) we get
\[
n^2q^2 < Dq^2 < (n+1)^2q^2. \quad (3)
\]
Upon substituting from (2) for the middle term in (3) and then removing the squares, we find:
\[
nq < p < (n+1)q \quad (4)
\]
Let $s = p - nq$ then it follows immediately from (4) that $0 < s < q$.

Furthermore, upon multiplying through (1) by $p^2$ we find:

$$n^2p^2 < Dp^2 < (n+1)^2p^2$$

(5)

Now substituting: $p^2 = Dq^2$ for the middle term in (5) and removing the squares we get:

$$np < Dq < (n+1)p$$

(6)

Let $r = Dq - np$ then it follows immediately from (6) that $0 < r < p$.

Now consider the difference:

$$Ds^2 - r^2 = D(p-nq)^2 - (Dq-np)^2$$

$$= Dp^2 - 2Dnpq + Dn^2q^2 - D^2q^2 + 2Dnpq - n^2p^2$$

$$= Dp^2 + Dn^2q^2 - D^2q^2 - n^2p^2$$

Using the fact that $Dq^2 = p^2$ it is evident that the right hand side reduces to zero.

Thus,

$$Ds^2 - r^2 = 0 \implies D = \frac{r^2}{s^2} \implies \sqrt{D} = \frac{r}{s}$$

But $s$ is greater than zero and less than $q$ which contradicts the assumption (*) about $q$. So now the result follows by contradiction.

THEOREM 3: The decimal expansion of any irrational number never repeats or terminates.

To show this, suppose we divide integers $p$ by $q$ (where $q$ is nonzero). When long division is applied to the division of $p$ by $q$, only $q$ remainders are possible (namely, 0, 1, 2, . . . , $q-1$).

If 0 appears as a remainder, then decimal expansion terminates.

If 0 never occurs, then the algorithm can run at most $q-1$ steps without using any remainder more than once. After that, a remainder must recur, and then the decimal expansion repeats.

It is not difficult to show that every recurring decimal represents a rational number.

**Exercise:** Express the number $A = 0.7162162162 \ldots$ as the quotient of two integers.