Ptolemy's Theorem

**Ptolemy’s Theorem** is a relation in Euclidean geometry between the four sides and two diagonals of a **cyclic quadrilateral** (i.e., a quadrilateral whose vertices lie on a common circle). The theorem is named after the Greek astronomer and mathematician Ptolemy (Claudius Ptolemaeus).

If the quadrilateral is given with its four vertices $A$, $B$, $C$, and $D$ in order, then the theorem states that:

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|$$

This relation may be verbally expressed as follows:

*If a quadrilateral is inscribed in a circle then the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals.*

**Proof**

**Geometric proof of Ptolemy’s Theorem**

1. Let $ABCD$ be a cyclic quadrilateral.
2. Note that on the chord $BC$, the inscribed angles $\angle BAC = \angle BDC$,
   and on $AB$, $\angle ADB = \angle ACB$.
3. Construct $K$ on $AC$ such that $\angle ABK = \angle CBD$;

   [Note that: $\angle ABK + \angle CBK = \angle ABC = \angle CBD + \angle ABD \Rightarrow \angle CBK = \angle ABD$.]

4. Now, by common angles $\triangle ABK$ is similar to $\triangle DBC$, and likewise $\triangle KBC \sim \triangle ABD$. 
5. Thus, \( \frac{|AK|}{|AB|} = \frac{|DC|}{|DB|} \) and \( \frac{|KC|}{|BC|} = \frac{|AD|}{|BD|} \) due to the similarities noted above:

\[
\begin{align*}
\triangle ABK & \sim \triangle DBC \\
\triangle KBC & \sim \triangle ABD
\end{align*}
\]

1. So \(|AK| \cdot |DB| = |AB| \cdot |DC|\), and \(|KC| \cdot |BD| = |BC| \cdot |AD|\);
2. Adding, \(|AK| \cdot |DB| + |KC| \cdot |BD| = |AB| \cdot |DC| + |BC| \cdot |AD|\);
3. Equivalently, \((|AK| + |KC|) \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|\);
4. But \(|AK| + |KC| = |AC|\), so
5. \(|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |DA|\); Q.E.D.

Some Corollaries to Ptolemy’s Theorem

CORROLARY 1:

Given an equilateral triangle \( \triangle ABC \) inscribed in a circle and a point \( Q \) on the circle. Then the distance from point \( Q \) to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.

In the figure, it follows that

\[
q = p + r
\]

CORROLARY 2:

In any regular pentagon the ratio of the length of a diagonal to the length of a side is the golden ratio, \( \varphi \).

In the figure, it follows that

\[
\varphi = \frac{b}{a}
\]