The Laws of Sines and Cosines

I. The Law of Sines

We have already seen that \( \triangle ABC \) with the acute angle \( \alpha \) has area:

\[
\text{Area}(\triangle ABC) = \frac{1}{2} \cdot b \cdot c \cdot \sin \alpha
\]

In case \( \gamma \) is obtuse, then we have

\[
\text{Area}(\triangle ABC) = \frac{1}{2} \cdot b \cdot h \quad \text{where}
\]

\[
\sin (180^\circ - \gamma) = \frac{h}{a} \quad \Rightarrow \quad h = a \cdot \sin (180^\circ - \gamma)
\]

\[
\Rightarrow \quad h = a \cdot \sin \gamma \quad \text{by the Supplementary Angle Identity.}
\]

So

\[
\text{Area}(\triangle ABC) = \frac{1}{2} \cdot a \cdot b \cdot \sin \gamma.
\]

Thus, the formula:

\[
\text{Area of Triangle} = \frac{1}{2} \times \left( \text{product of length of any two sides} \right) \times \left( \text{sine of included angle} \right)
\]

is valid in all cases. That is,

\[
\text{Area}(\triangle ABC) = \frac{1}{2} \cdot b \cdot c \cdot \sin \alpha = \frac{1}{2} \cdot a \cdot c \cdot \sin \beta = \frac{1}{2} \cdot a \cdot b \cdot \sin \gamma
\]

Multiply each side of the above equality by 2 to obtain

\[
b \cdot c \cdot \sin \alpha = a \cdot c \cdot \sin \beta = a \cdot b \cdot \sin \gamma.
\]

Now divide each side by \( abc \) to arrive at the triangle identity:

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}
\]

This formula is known as the Law of Sines.

Notice that the law of sines can be written in the alternative form:

\[
\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}
\]
Solving a Triangle (AAS)

If two angles of the triangle $\triangle ABC$ are given, then the third angle can be found by using the relationship:

$$\alpha + \beta + \gamma = 180^\circ ;$$

hence, the three denominators $\sin \alpha$, $\sin \beta$, and $\sin \gamma$ can be found using a calculator. Now, if any one of the sides $a$, $b$, or $c$ is also given, then the equations

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

can be solved for the remaining two sides.

The following example indicates the procedure for solving a triangle when two angles and one side are given (or can be determined from the information provided.) Unless otherwise indicated, we shall round off all angles to the nearest hundredth of a degree, and all side lengths to four significant digits.

**Example 1**

In $\triangle ABC$, suppose that $\alpha = 41^\circ$, $\beta = 77^\circ$, and $a = 74$. Solve for $\gamma$, $b$, and $c$. 

![Diagram of triangle ABC with angles and side labels]

Solution  

$\gamma = 180^\circ - 41^\circ - 77^\circ = 62^\circ$

By the law of sines,  

$$\frac{a}{\sin 41^\circ} = \frac{b}{\sin 77^\circ} = \frac{c}{\sin 62^\circ} ;$$

hence, since $a = 74$,  

$$b = \frac{a \sin 77^\circ}{\sin 41^\circ} = \frac{74 \sin 77^\circ}{\sin 41^\circ} = 109.9$$

and  

$$c = \frac{a \sin 62^\circ}{\sin 41^\circ} = \frac{74 \sin 62^\circ}{\sin 41^\circ} = 99.59 .$$

Note that in this case there is always a unique solution by the Angle-Side-Angle criteria for congruent triangles.

Solving a Triangle (SSA)

Because there are several possibilities, the situation in which you are given the lengths of two sides of a triangle and the angle opposite one of them is called the ambiguous case.
For instance, suppose you are given side \( a \), side \( b \), and angle \( \alpha \) in \( \Delta ABC \). You might try to construct \( \Delta ABC \) from this information by drawing a line segment \( \overline{AC} \) of length \( b \) and a ray \( l \) that starts at \( A \) and makes an angle \( \alpha \) with \( \overline{AC} \) (Figure 1). To find the remaining vertex \( B \), you could use a compass to draw an arc of a circle of radius \( a \) with center \( C \). If the arc intersects the ray \( l \) at point \( B \), then \( \Delta ABC \) is the desired triangle.

![Figure 1](image1)

As figure 2 illustrates, there are actually four possibilities if you try to construct \( \Delta ABC \) by the above method in case \( \alpha \) is acute:

(i) The circle does not intersect the ray \( l \) at all and there is no triangle \( \Delta ABC \) (Figure 2a).

(ii) The circle intersects the ray \( l \) in exactly one point \( B \) and there is just one right triangle \( \Delta ABC \) (Figure 2b).

(iii) The circle intersects the ray \( l \) in two points \( B_1 \) and \( B_2 \) and there are two triangles \( \Delta AB_1C \) and \( \Delta AB_2C \) (Figure 2c).

(iv) The circle intersects the ray \( l \) in exactly one point \( B \) and there is just one acute triangle \( \Delta ABC \) (Figure 2d).

![Figure 2](image2)

(a) \( a < b \sin \alpha \Rightarrow \) no triangle possible

(b) \( a = b \sin \alpha \Rightarrow \) one right triangle
\( \alpha \) is obtuse, then there are only two possibilities as shown in Figure 3.

**Figure 3**

(\( a \)) \( a > b \) \( \Rightarrow \) one triangle

(\( b \)) \( b \) \( \Rightarrow \) no triangle possible

In the ambiguous case, you can always use a calculator to solve the triangle \( \triangle ABC \). Just use the law of sines,

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}
\]

to evaluate \( \sin \beta \):

\[
\sin \beta = b \left( \frac{\sin \alpha}{a} \right), \quad 0 < \beta < 180^\circ.
\]
Recall that the sine of an angle is never greater than 1; hence, if \( b \left( \frac{\sin \alpha}{a} \right) > 1 \), then this trigonometric equation has no solution, and no triangle satisfies the given conditions. If \( b \left( \frac{\sin \alpha}{a} \right) = 1 \), then the equation has only one solution, \( \beta = 90^\circ \). If \( b \left( \frac{\sin \alpha}{a} \right) < 1 \), then the trigonometric equation has two solutions. Namely,

\[
\beta_1 = \arcsin \left( \frac{b \sin \alpha}{a} \right) \quad \text{and} \quad \beta_2 = 180^\circ - \beta_1 .
\]

Once you have determined \( \beta \) (or \( \beta_1 \) and \( \beta_2 \)), you know two angles and two sides of the triangle (or triangles), and you can solve the triangle by using the methods previously explained. Even if there are two solutions \( \beta_1 \) and \( \beta_2 \) of the trigonometric equation for \( \beta \), it is possible that only one of these solutions corresponds to an actual triangle satisfying the given conditions (see Figure 2d and Figure 3a).

**Example 2**

Solve the triangle in each case.

1. \( \alpha = 30^\circ, \ a = 8, \ b = 5 \).

Since \( \alpha \) is acute and \( a \geq b \) there is only one acute triangle constructible.

\[
\frac{\sin \beta}{5} = \frac{\sin 30^\circ}{8} \Rightarrow \sin \beta = \left( 5 \left( \frac{1}{2} \right) \right) = \frac{5}{16}
\]

\[
\Rightarrow \beta = \arcsin \left( \frac{5}{16} \right) \approx 18.21^\circ .
\]

Then \( \gamma = 180^\circ - 30^\circ - 18.21^\circ = 131.79^\circ \).

Finally, \( \frac{\sin 131.79^\circ}{c} = \frac{\sin 30^\circ}{8} \Rightarrow c = 8 \left( \frac{\sin 131.79^\circ}{\sin 30^\circ} \right) \approx 11.93 \).

2. \( \alpha = 30^\circ, \ a = 5, \ b = 8 \).

Here \( \alpha \) is acute and \( b \sin a = \frac{8}{2} < a < b \)

\[
\Rightarrow \text{there are two triangles constructible.}
\]

\[
\sin 30^\circ = \frac{\sin \beta}{8} \Rightarrow \sin \beta = \left( 8 \left( \frac{1}{2} \right) \right) = \frac{4}{5} .
\]
Thus, \( \beta_1 = \arcsin \left( \frac{4}{5} \right) = 53.13^\circ \). Then \( \gamma_1 = 180^\circ - 30^\circ - 53.13^\circ = 96.87^\circ \) and

\[
\frac{\sin 30^\circ}{5} = \frac{\sin 96.87^\circ}{c} \Rightarrow c = 5 \left( \frac{\sin 96.87^\circ}{\sin 30^\circ} \right) = 9.928
\]

To find the measure of \( \beta_2 \) observe that

\[
\sin \beta_2 = \sin (180^\circ - \beta_1) = \sin \beta_1.
\]

Thus, \( \beta_2 = 180^\circ - \beta_1 = 180^\circ - 53.13^\circ = 126.87^\circ \)
and \( \gamma_2 = 180^\circ - 30^\circ - 126.87^\circ = 23.13^\circ \).

Then \( \frac{\sin 30^\circ}{5} = \frac{\sin 23.13^\circ}{c} \Rightarrow c = 5 \left( \frac{\sin 23.13^\circ}{\sin 30^\circ} \right) = 3.928 \).

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II. The Law of Cosines

When two sides and the included angle (SAS) or three sides (SSS) of a triangle are given, we cannot apply the law of sines to solve the triangle. In such cases, the law of cosines may be applied.

**Theorem: The Law of Cosines**

In the general triangle \( \triangle ABC \), the square of the length of any side is equal to the sum of the squares of the lengths of the other two sides minus twice the product of those side lengths times the cosine of the angle between them.

\[
c^2 = a^2 + b^2 - 2ab \cos \gamma
\]

\[
b^2 = a^2 + c^2 - 2ac \cos \beta
\]

\[
a^2 = b^2 + c^2 - 2bc \cos \alpha
\]

To prove the theorem, we place triangle \( \triangle ABC \) in a coordinate plane with vertices labeled counterclockwise and so that one side lies on the positive \( x \) axis and one vertex is at \( O \).

Suppose that \( A \) is at \((0, 0)\). Then \( B = (c, 0) \) and \( C = (b \cos \alpha, b \sin \alpha) \).

Thus, \( \|BC\|^2 = (b \cos \alpha - c)^2 + (b \sin \alpha)^2 = a^2 \).

\[
b^2 \cos^2 \alpha - 2bc \cos \alpha + c^2 + b^2 \sin^2 \alpha = a^2.
\]
So \[ a^2 = b^2 + c^2 - 2bc \cos \alpha. \]

Now rotate the triangle so that \( B \) is at the origin and \( C \) is on the positive \( x \) axis.

An analogous argument now gives

\[ b^2 = a^2 + c^2 - 2ac \cos \beta. \]

When \( C \) is at the origin, we find

\[ c^2 = a^2 + b^2 - 2ab \cos \gamma. \]

**Example 3**

**SAS case:** Solve the triangle \( \triangle ABC \) if \( \alpha = 60^\circ \), \( b = 14 \), \( c = 10 \).

Since \[ a^2 = b^2 + c^2 - 2bc \cos \alpha \]
\[ = (14)^2 + (10)^2 - 2(14)(10)\cos(60^\circ) \]
\[ = 196 + 100 - 140 \]
\[ a^2 = 156 \]
\[ \Rightarrow a = 12.49. \]

It is geometrically evident that \( \beta \) is acute and by the law of sines

\[ \frac{\sin 60^\circ}{12.49} = \frac{\sin \beta}{14} \Rightarrow \sin \beta = \left( \frac{14}{12.49} \right) \left( \frac{\sqrt{3}}{2} \right) = \frac{7\sqrt{3}}{12.49} \]

\[ \Rightarrow \beta = \arcsin \left( \frac{7\sqrt{3}}{12.49} \right) = 76.10^\circ. \]

Then \( \gamma = 180^\circ - 60^\circ - 76.10^\circ = 43.90^\circ. \)
Example 4

SSS case: Solve the triangle $\triangle ABC$ if $a = 5$, $b = 6$, $c = 7$.

\[
a^2 = b^2 + c^2 - 2bc \cos \alpha \\
25 = 36 + 49 - 2(6)(7) \cos \alpha \\
25 = 85 - 84 \cos \alpha \Rightarrow \cos \alpha = \frac{60}{84} = \frac{5}{7} \\
\Rightarrow \alpha = \arccos \left( \frac{5}{7} \right) = 44.42^\circ
\]

Note: there is no other angle $\theta$ for which: $0^\circ \leq \theta < 180^\circ$ and $\cos \theta = \frac{5}{7}$.

Then by the law of sines

\[
\frac{\sin 44.42^\circ}{5} = \frac{\sin \beta}{6} \Rightarrow \sin \beta = \left( \frac{6}{5} \right) \sin 44.42^\circ
\]

\[
\Rightarrow \beta = \arcsin \left( \left( \frac{6}{5} \right) \sin 44.42^\circ \right) = 57.13^\circ . \text{ Clearly, } \beta \text{ must be acute.}
\]

Then $\gamma = 180^\circ - 44.42^\circ - 57.13^\circ = 78.45^\circ$.

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**Theorem: Heron’s Area Formula**

The area of a triangle with sides $a$, $b$ and $c$ and semiperimeter $s = \frac{a+b+c}{2}$ has area $A$ given by

\[
A = \sqrt{s(s-a)(s-b)(s-c)} .
\]

The proof follows from the law of cosines expressed in the form:

\[
2\ bc \cdot \cos \alpha = b^2 + c^2 - a^2
\]

Note that $A = \frac{1}{2} \ ch = \frac{1}{2} \ bc \sin \alpha \Rightarrow A^2 = \frac{1}{4} b^2 c^2 \sin^2 \alpha$. 

Now we may obtain the desired formula by algebraic manipulation.

\[
A^2 = \frac{1}{4} b^2 c^2 \sin^2 \alpha = \frac{1}{4} b^2 c^2 (1 - \cos^2 \alpha)
\]

\[
= \frac{1}{16} (2bc)(1 + \cos \alpha) (2bc)(1 - \cos \alpha)
\]

\[
= \frac{1}{16} (2bc + 2bc \cdot \cos \alpha)(2bc - 2bc \cdot \cos \alpha)
\]

\[
= \frac{1}{16} (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)
\]

\[
= \left( \frac{1}{16} \right) \left[ (b + c)^2 - a^2 \right] \left[ a^2 - (b - c)^2 \right]
\]

\[
= \frac{(b+c+a)}{2} \cdot \frac{(b+c-a)}{2} \cdot \frac{(a-b+c)}{2} \cdot \frac{(a+b-c)}{2}
\]

\[
= \left[ \frac{a+b+c}{2} \right] \cdot \left[ \frac{a+b+c}{2} - a \right] \cdot \left[ \frac{a+b+c}{2} - b \right] \cdot \left[ \frac{a+b+c}{2} - c \right]
\]

\[
A^2 = s(s-a)(s-b)(s-c).
\]