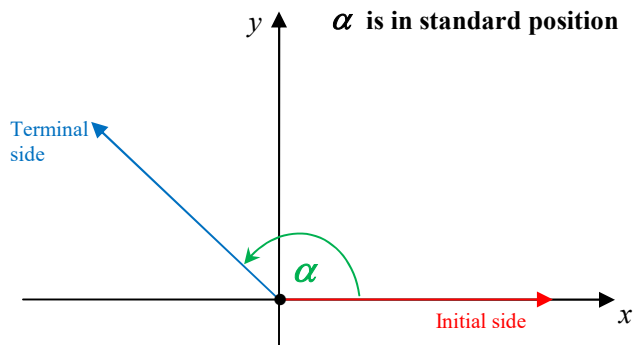


## 2. TRIGONOMETRIC FUNCTIONS OF GENERAL ANGLES

In order to extend the definitions of the six trigonometric functions to general angles, we shall make use of the following ideas: In a Cartesian coordinate system, *an angle  $\alpha$  is said to be in standard position if its vertex is at the origin  $O$  and its initial side coincides with the positive  $x$  axis* (Figure 2.1). An angle is said to be in a certain quadrant if, when the angle is in standard position, the terminal side lies in that quadrant. For instance, a  $65^\circ$  angle **lies in quadrant I** or is simply said to be a **quadrant I angle**. As Figure 2.2 b shows, an angle of measure  $-187^\circ$  is a **quadrant II angle**.

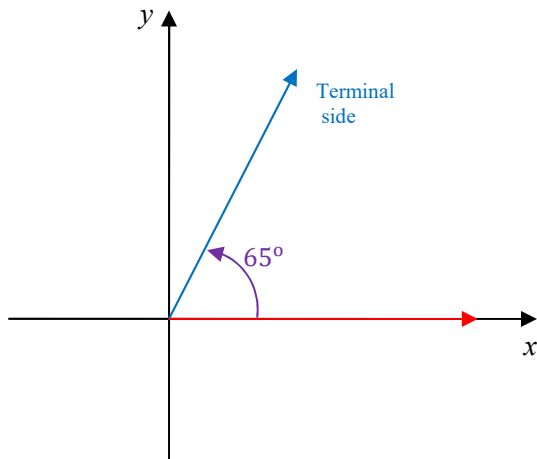
If the terminal side of an angle in standard position lies along either the  $x$  axis or the  $y$  axis, then the angle is called **quadrantal**. For example,  $-360^\circ$ ,  $-270^\circ$ ,  $-180^\circ$ ,  $-90^\circ$ ,  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ ,  $360^\circ$  are all quadrantal angles. Evidently, an angle is quadrantal if and only if its measure is an integer multiple of  $90^\circ$  (or  $\frac{\pi}{2}$  radians).

**Figure 2.1**

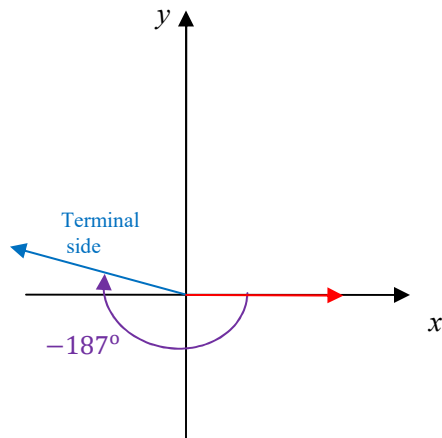


**Figure 2.2**

**(a) quadrant I angle**



**(b) quadrant II angle**

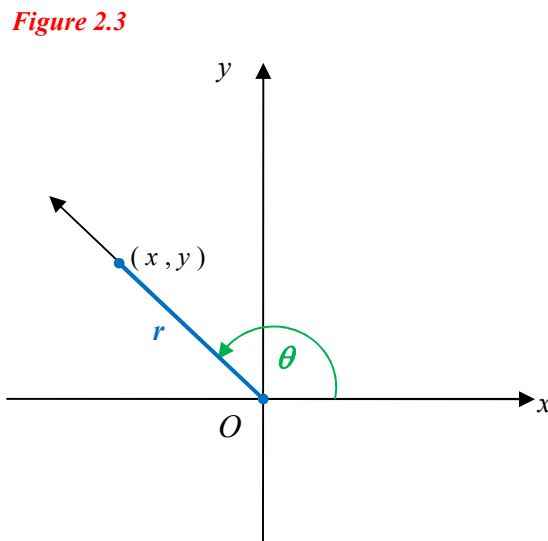


### Definition 2.1: Trigonometric Functions of a General Angle

Let  $\theta$  be an angle in standard position and suppose that  $(x, y)$  is any point other than  $(0, 0)$  on the terminal side of  $\theta$  (Figure 2.3). If  $r = \sqrt{x^2 + y^2}$  is the distance between  $(x, y)$  and  $(0, 0)$ , then the six trigonometric functions of  $\theta$  are defined by

$$\begin{aligned}\sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y}\end{aligned}$$

provided that the denominators are not zero.



Using similar triangles, you can see that the values of the six trigonometric functions in Definition 2.1 depend only on the angle  $\theta$  and not on the choice of the point  $(x, y)$  on the terminal side of  $\theta$ .

### Example 2.1

Evaluate the six trigonometric functions of the angle  $\theta$  in standard position if the terminal side of  $\theta$  contains the point  $(x, y) = (2, -1)$ .

Here,  $x = 2$ ,  $y = -1$ , and

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

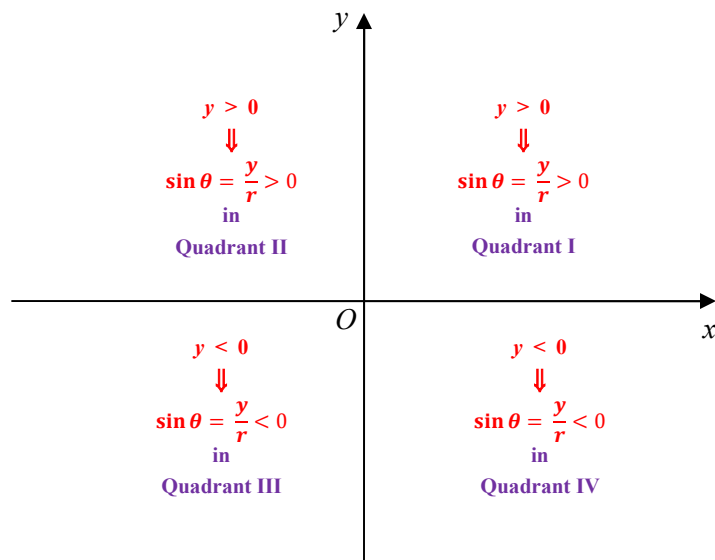
Thus,

$$\begin{aligned}\sin \theta &= \frac{y}{r} = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5} & \csc \theta &= \frac{r}{y} = \frac{\sqrt{5}}{-1} = -\sqrt{5} \\ \cos \theta &= \frac{x}{r} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} & \sec \theta &= \frac{r}{x} = \frac{\sqrt{5}}{2} \\ \tan \theta &= \frac{y}{x} = \frac{-1}{2} = -\frac{1}{2} & \cot \theta &= \frac{x}{y} = \frac{2}{-1} = -2.\end{aligned}$$

You can determine the algebraic signs of the trigonometric functions for angles in the various quadrants by recalling the algebraic signs of  $x$  and  $y$  in these quadrants and

remembering that  $r$  is always positive. For instance, as Figure 2.4 shows,  $\sin \theta = \frac{y}{r}$  is positive in quadrants I and II (where both  $y$  and  $r$  are positive), and it is negative in quadrants III and IV (where  $y$  is negative and  $r$  is positive). By proceeding in a similar way, you can determine the signs of the remaining trigonometric functions in the various quadrants and thus confirm the results in Table 2.1.

**Figure 2.4**



**Table 2.1**

Quadrant Containing $\theta$	Positive Functions	Negative Functions
I	All	None
II	$\sin \theta, \csc \theta$	$\cos \theta, \sec \theta, \tan \theta, \cot \theta$
III	$\tan \theta, \cot \theta$	$\sin \theta, \csc \theta, \cos \theta, \sec \theta$
IV	$\cos \theta, \sec \theta$	$\sin \theta, \csc \theta, \tan \theta, \cot \theta$

**Example 2.2**

Find the quadrant in which  $\theta$  lies if  $\tan \theta > 0$  and  $\sin \theta < 0$ .

This example can be worked by using Table 2.1; however, rather than relying on the table, we prefer to reason as follows: Let  $(x, y)$  be a point other than the origin on the terminal side of  $\theta$  (in standard position). Because  $\tan \theta = \frac{y}{x} > 0$ , we see that  $x$  and  $y$  have the same algebraic sign. Furthermore, since  $\sin \theta = \frac{y}{r} < 0$ , it follows that  $y < 0$ . Because  $x < 0$  and  $y < 0$ , the angle is in quadrant III.

## Reciprocal Identities

If  $\theta$  is an angle for which the functions are defined, then:

$$(i) \csc \theta = \frac{1}{\sin \theta} \quad (ii) \sec \theta = \frac{1}{\cos \theta} \quad (iii) \cot \theta = \frac{1}{\tan \theta} .$$

## Quotient Identities

If  $\theta$  is an angle for which the functions are defined, then:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{and} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} .$$

### Example 2.3

If  $\sin \theta = \frac{1}{3}$  and  $\cos \theta = \frac{-2\sqrt{2}}{3}$ , find the values of the other four trigonometric functions of  $\theta$ .

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\frac{1}{3}}{\frac{-2\sqrt{2}}{3}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{-2\sqrt{2}}{3}} = -\frac{3}{2\sqrt{2}} = -\frac{3\sqrt{2}}{4}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{-\frac{\sqrt{2}}{4}} = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{1}{3}} = 3.$$

By using the reciprocal and quotient identities, you can quickly recall the algebraic signs of the secant, cosecant, tangent, and cotangent in the four quadrants (Table 1), if you know the algebraic signs of the sine and cosine in these quadrants.

Another important identity is derived as follows: Again suppose that  $\theta$  is an angle in standard position and that  $(x, y)$  is a point other than the origin on the terminal side of  $\theta$  (Figure 9). Because  $r = \sqrt{x^2 + y^2}$ , we have  $x^2 + y^2 = r^2$ , so

$$(\cos \theta)^2 + (\sin \theta)^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

The relationship:  $(\cos \theta)^2 + (\sin \theta)^2 = 1$

is called the **fundamental Pythagorean identity** because its derivation involves the fact that  $x^2 + y^2 = r^2$ , which is a consequence of the Pythagorean theorem.

The fundamental Pythagorean identity is used quite often, and it would be bothersome to write the parentheses each time for  $(\cos \theta)^2$  and  $(\sin \theta)^2$ ; yet, if the parentheses were simply omitted, the resulting expressions would be misunderstood. (For instance,  $\cos \theta^2$  is usually understood to mean the cosine of the square of  $\theta$ .) Therefore, it is customary to write  $\cos^2 \theta$  and  $\sin^2 \theta$  to mean  $(\cos \theta)^2$  and  $(\sin \theta)^2$ . Similar notation is used for the remaining trigonometric functions and for powers other than 2. Thus,  $\cot^4 \theta$  means  $(\cot \theta)^4$ ,  $\sec^n \theta$  means  $(\sec \theta)^n$ , and so forth. With this notation, the fundamental Pythagorean identity becomes

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Actually, there are three Pythagorean identities – the fundamental identity and two others derived from it.

### Pythagorean Identities

If  $\theta$  is an angle for which the functions are defined, then:

(i)  $\cos^2 \theta + \sin^2 \theta = 1$       (ii)  $1 + \tan^2 \theta = \sec^2 \theta$       (iii)  $1 + \cot^2 \theta = \csc^2 \theta$

We already proved (i). To prove (ii), we divide both sides of (i) by  $\cos^2 \theta$  to obtain

$$1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

or

$$1 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 = \left(\frac{1}{\cos \theta}\right)^2,$$

provided that  $\cos \theta \neq 0$ . Since

$$\frac{\sin \theta}{\cos \theta} = \tan \theta \quad \text{and} \quad \frac{1}{\cos \theta} = \sec \theta,$$

we have that

$$1 + \tan^2 \theta = \sec^2 \theta.$$

Identity (iii) is proved by dividing both sides of (i) by  $\sin^2 \theta$ .

### **Example 2.4**

The value of one of the trigonometric functions of an angle  $\theta$  is given along with the information about the quadrant in which  $\theta$  lies, Find the values of the other five trigonometric functions of  $\theta$ :

(a)  $\sin \theta = \frac{5}{13}$ ,  $\theta$  in quadrant II.

By the fundamental Pythagorean identity,  $\cos^2 \theta + \sin^2 \theta = 1$ , so

$$\cos^2 \theta = 1 - \sin^2 \theta = 1 - \left(\frac{5}{13}\right)^2 = 1 - \frac{25}{169} = \frac{169-25}{169} = \frac{144}{169}.$$

$$\text{Therefore, } \cos \theta = \pm \sqrt{\frac{144}{169}} = \pm \frac{12}{13}.$$

Because  $\theta$  is in quadrant II, we know that  $\cos \theta$  is negative; hence,

$$\cos \theta = -\frac{12}{13}.$$

It follows that

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{5/13}{-12/13} = -\frac{5}{12} & \cot \theta &= \frac{1}{\tan \theta} = \frac{1}{-5/12} = -\frac{12}{5} \\ \sec \theta &= \frac{1}{\cos \theta} = \frac{1}{-12/13} = -\frac{13}{12} & \csc \theta &= \frac{1}{\sin \theta} = \frac{1}{5/13} = \frac{13}{5} \end{aligned}$$

(b)  $\tan \theta = -\frac{8}{7}$  and  $\sin \theta < 0$ .

Because  $\tan \theta < 0$  only in quadrants II and IV, and  $\sin \theta < 0$  only in quadrants III and IV, it follows that  $\theta$  must be in quadrant IV. By part (ii)  $\sec^2 \theta = 1 + \tan^2 \theta$ , so

$$\sec \theta = \pm \sqrt{1 + \tan^2 \theta} = \pm \sqrt{1 + \left(-\frac{8}{7}\right)^2} = \pm \sqrt{1 + \frac{64}{49}} = \pm \sqrt{\frac{113}{49}} = \pm \frac{\sqrt{113}}{7}.$$

Since  $\theta$  is in quadrant IV,  $\sec \theta > 0$ ; hence,

$$\sec \theta = \frac{\sqrt{113}}{7}.$$

$$\text{Because } \sec \theta = \frac{1}{\cos \theta},$$

it follows that

$$\cos \theta = \frac{1}{\sec \theta} = \frac{1}{\sqrt{113}/7} = \frac{7}{\sqrt{113}} = \frac{7\sqrt{113}}{113}.$$

$$\text{Now, } \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\text{so } \sin \theta = (\tan \theta)(\cos \theta) = \left(-\frac{8}{7}\right) \left(\frac{7\sqrt{113}}{113}\right) = -\frac{8\sqrt{113}}{113}.$$

$$\text{Finally, } \csc \theta = \frac{1}{\sin \theta} = \frac{1}{-8\sqrt{113}/113} = -\frac{\sqrt{113}}{8}$$

$$\text{and } \cot \theta = \frac{1}{\tan \theta} = \frac{1}{-8/7} = -\frac{7}{8}.$$

In the applications of trigonometry, and especially in calculus, it is often necessary to make trigonometric calculations, as we have done in this section, *without the use of calculators or tables*.

### **Section 2 Problems**

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In problems 1 to 10, sketch two coterminal angles  $\alpha$  and  $\beta$  in standard position whose terminal side contains the given point. Arrange it so that  $\alpha$  is positive,  $\beta$  is negative, and neither angle exceeds one revolution. In each case, name the quadrant in which the angle lies, or indicate that the angle is quadrantal.

- |                                   |                              |
|-----------------------------------|------------------------------|
| 1. $(1, 2)$                       | 2. $(2, -\frac{4}{3})$       |
| 3. $(-5, 0)$                      | 4. $(-3, -4)$                |
| 5. $(5, -3)$                      | 6. $(0, 4)$                  |
| 7. $(-1, 1)$                      | 8. $(\frac{\sqrt{2}}{2}, 0)$ |
| 9. $(-\frac{3}{4}, -\frac{3}{4})$ | 10. $(0, -3)$                |

In problems 11 to 18, specify and sketch three angles that are coterminal with the given angle in standard position.

- |                      |                       |
|----------------------|-----------------------|
| 11. $60^\circ$       | 12. $-15^\circ$       |
| 13. $-\frac{\pi}{4}$ | 14. $\frac{4\pi}{3}$  |
| 15. $-612^\circ$     | 16. $-\frac{7\pi}{4}$ |
| 17. $\frac{5\pi}{6}$ | 18. $1440^\circ$      |

In problems 19 to 28, evaluate the six trigonometric functions of the angle  $\theta$  in standard position if the terminal side of  $\theta$  contains the given point  $(x, y)$ . [Do not use a calculator – leave all answers in the form of a fraction or an integer.] In each case, sketch one of the coterminal angles  $\theta$ .

- |                              |                |
|------------------------------|----------------|
| 19. $(4, 3)$                 | 20. $(2, 7)$   |
| 21. $(-5, 12)$               | 22. $(-2, 4)$  |
| 23. $(-3, -4)$               | 24. $(1, 1)$   |
| 25. $(7, 3)$                 | 26. $(1, -3)$  |
| 27. $(-\sqrt{3}, -\sqrt{2})$ | 28. $(20, 21)$ |

29. Is there any angle  $\theta$  for which  $\sin \theta = \frac{5}{4}$ ? Explain.

- 30.** Using similar triangles, show that the values of the six trigonometric functions in Definition 1 depend only on the angle  $\theta$  and not on the choice of the point  $(x, y)$  on the terminal side of  $\theta$ .
- 31.** In each case, assume that  $\theta$  is an angle in standard position and find the quadrant in which it lies.
- (a)  $\tan \theta > 0$  and  $\sec \theta > 0$
  - (b)  $\sin \theta > 0$  and  $\sec \theta < 0$
  - (c)  $\sin \theta > 0$  and  $\cos \theta < 0$
  - (d)  $\sec \theta > 0$  and  $\tan \theta < 0$
  - (e)  $\tan \theta > 0$  and  $\csc \theta < 0$
  - (f)  $\cos \theta < 0$  and  $\csc \theta < 0$
  - (g)  $\sec \theta > 0$  and  $\cot \theta < 0$
  - (h)  $\cot \theta > 0$  and  $\sin \theta > 0$
- 32.** Is there any angle  $\theta$  for which  $\sin \theta > 0$  and  $\csc \theta < 0$ ? Explain.
- 33.** Give the algebraic sign of each of the following.
- (a)  $\cos 163^\circ$
  - (b)  $\sin 211^\circ$
  - (c)  $\sec(-355^\circ)$
  - (d)  $\tan\left(\frac{7\pi}{4}\right)$
  - (e)  $\cot\left(-\frac{28\pi}{45}\right)$
  - (f)  $\csc\left(-\frac{5\pi}{12}\right)$
  - (g)  $\sec\left(\frac{38\pi}{15}\right)$
- 34.** If  $\theta$  is an angle for which the functions are defined, show that  $\sec \theta - (\sin \theta)(\tan \theta) = \cos \theta$ .
- 35.** If  $\sin \theta = -\frac{5}{13}$  and  $\cos \theta = \frac{12}{13}$ , use the reciprocal and quotient identities to find
- (a)  $\sec \theta$
  - (b)  $\csc \theta$
  - (c)  $\tan \theta$
  - (d)  $\cot \theta$ .



36. If  $\sec \theta = -\frac{5}{4}$  and  $\csc \theta = -\frac{5}{3}$ , use the reciprocal and quotient identities to find  
(a)  $\sin \theta$       (b)  $\cos \theta$       (c)  $\tan \theta$       (d)  $\cot \theta$ .

In Problems 37 to 48, the values of one of the trigonometric functions of an angle  $\theta$  is given along with information about the quadrant (Q) in which  $\theta$  lies. Find the values of the other five trigonometric functions of  $\theta$ .

37.  $\sin \theta = \frac{3}{4}$ ,       $\theta$  in  $Q_I$   
38.  $\cos \theta = \frac{12}{13}$ ,       $\theta$  in  $Q_{IV}$   
39.  $\sin \theta = -\frac{3}{4}$ ,       $\theta$  in  $Q_{III}$   
40.  $\sin \theta = \frac{3}{4}$ ,       $\theta$  not in  $Q_I$   
41.  $\cos \theta = \frac{4}{7}$ ,       $\sin \theta < 0$   
42.  $\cos \theta = \frac{1}{3}$ ,       $\theta$  not in  $Q_I$   
43.  $\csc \theta = \frac{3}{2}$ ,       $\theta$  in  $Q_I$   
44.  $\sec \theta = -\frac{5}{3}$ ,       $\theta$  in  $Q_{III}$   
45.  $\tan \theta = \frac{4}{3}$ ,       $\theta$  in  $Q_I$   
46.  $\tan \theta = 2$ ,       $\sin \theta < 0$   
47.  $\cot \theta = -\frac{12}{5}$ ,       $\csc \theta > 0$   
48.  $\csc \theta = -\frac{25}{7}$ ,       $\sec \theta < 0$