Volumes of Cylinders, Pyramids, Cones & Spheres

The volume of a three-dimensional body is a numerical characteristic of the body; in the simplest case, when the body can be decomposed into a finite set of unit cubes (i.e. cubes with edges of unit length), it is equal to the number of these cubes. Volumes of three-dimensional bodies (i.e. sets in three-dimensional Euclidean space) for which volume can be defined have properties analogous to those of areas of plane figures:

1) volume is non-negative;
2) volume is additive: If two bodies $P$ and $Q$ with no common interior points have volumes $v(P)$ and $v(Q)$, then the volume of their union is the sum of their volumes, $v(P \cup Q) = v(P) + v(Q)$;
3) volume is invariant with respect to displacements: If the volumes of bodies $P$ and $Q$ are defined and the bodies are congruent, then $v(P) = v(Q)$;
4) the volume of the unit cube is equal to one.

These properties imply that volume is monotone: If the volumes $v(P)$ and $v(Q)$ are defined for bodies $P$ and $Q$ and $P \subseteq Q$, then $v(P) \leq v(Q)$, and the volumes of two similar bodies are proportional to the cube of the factor of the corresponding similarity.

Cavalieri's Principle

The volumes of two objects are equal if the areas of their corresponding cross-sections are in all cases equal.

Bonaventura Francesco Cavalieri (1598 – 1647), was an Italian mathematician who developed a "method of the indivisibles," which he used to determine areas and volumes. It was a significant step on the way to modern infinitesimal calculus. His work was motivated by the classical Greek method of exhaustion which is a method for finding the area of a shape by inscribing inside it a sequence of polygons whose areas converge to the area of the containing shape. If the sequence is correctly constructed, the difference in area between the $n$th polygon and the containing shape will become arbitrarily small as $n$ becomes large. As this difference becomes arbitrarily small, the possible values for the area of the shape are systematically "exhausted" by the lower bound areas successively established by the sequence members. The idea originated with Antiphon, although it is not entirely clear how well he understood it. The theory was made rigorous by Eudoxus.

The works of Archimedes (in particular, his Message to Eratosthenes) indicate that — prior to Archimedes' logically precise method for estimating areas and volumes with the aid of sums of a very large number of terms that are decreasing without limit (i.e. infinitesimals in the modern sense of the word) — there also existed a more primitive, but more illustrative method, attributable to Democritus (4th century B.C.). It is pointed out by Archimedes, in particular, that Democritus determined the volume of the pyramid prior to Eudoxus (even though he failed to give a rigorous proof of his results).
In deriving the volume of a pyramid, the main difficulty encountered by Euclid and by Eudoxus was to prove that two pyramids with equal heights and equal base areas have equal volumes. Euclid overcame this difficulty in his Elements by using the method of exhaustion.

As reported by Archimedes, the "atomistic" method for proving the above theorem used by Democritus may be described as follows. Similarity considerations indicate that the cross-sectional areas of the pyramids are equal at equal heights; the volume of the pyramids are simply considered as the "sums" of these areas; hence, the equalities of the corresponding terms of the two sums prove that the sums themselves are equal as well. Archimedes quotes many examples of the use of this method in solving more complicated problems. Archimedes considered the method not as strict but as highly valuable heuristically (i.e. for arriving at new results, which must subsequently be more rigorously demonstrated); from our own point of view, this view was undoubtedly correct, since Democritus' method was merely an unfounded attempt to replace the process of passing to a limit

$$S = \lim_{n \to \infty} (\triangle_1^{(n)} + \triangle_2^{(n)} + \ldots + \triangle_n^{(n)})$$

by the invalid metaphysical hypothesis to the effect that volumes can be added.

By applying either Democritus' atomistic method or Cavalieri’s principle, we may conclude that the volume of a general cylinder, i.e. any three-dimensional body having identical cross-sections of constant area $A$ in each plane parallel to its base, is the same as that of the rectangular box having a square base of side length $\sqrt{A}$ and the same height as the given cylinder. Thus, the volume formula for the cylinder is:

$$V_{CYLINDER} = \text{area (base)} \cdot \text{height}$$

$$A_1 = A_2 = A_3$$

$$v(S_1) = v(S_2) = v(S_3)$$
This volume formula is intuitively clear for the case of the rectangular box having base area \( A \) square units and height \( h \), since it would require exactly \( A \) unit cubes to fully cover the base of the box and the cubes would be stacked in rows \( h \) cubes high in order to reach the top of the box. Thus, the rectangular box is composed of exactly \( A \cdot h \) unit cubes. I.e., it has volume \( A \cdot h \) cubic units.

**Theorem 1:** If a pyramid is formed by joining via line segments each point on a given parallelogram \( \square ABCD \) to a fixed point \( V \) not in the same plane as the parallelogram and the so formed pyramid: \( \square ABCD \oplus V \) is subsequently cut by a plane through the points \( V, B \) and \( D \), \( BD \) being a diagonal of the parallelogram, to form the two triangle based pyramids:

\[ \triangle ABD \oplus V \quad \text{and} \quad \triangle BCD \oplus V \]

then

\[ v (\triangle ABD \oplus V) = v (\triangle BCD \oplus V) \]

**Sketch of Proof:**
Since the diagonal of a parallelogram partitions it into two congruent triangles (ASA), and since in each plane parallel to the base parallelogram \( \square ABCD \), the cross section of the original pyramid has been partitioned along one of its diagonal by the cutting plane, then it follows that the two triangular based pyramids have equal areas in each of their corresponding cross-sections. Thus, due to Democritus’ atomistic’ method (or Cavalieri’s Principle), the two pyramids have equal volumes.

**Theorem 2:** A triangular prism can be divided into three equal pyramids, so that the volume of each one of the pyramids is one third of the volume of the prism that contains it.

**Proof:**
This theorem is Euclid’s Proposition 7 from Book 12 of the *Elements*.
We can prove it as follows:
If the triangular prism with the parallel triangular bases: \( \triangle ABC \) and \( \triangle DEF \) is cut by the planes:

\( \mathcal{P}_1 \) : through \( B, C \) and \( D \)

and

\( \mathcal{P}_2 \) : through \( C, D \), and \( E \)

then the prism is partitioned into three triangular pyramids.
Namely, 
\[ \triangle BDE \otimes C , \ \triangle ABD \otimes C , \text{ and } \triangle DEF \otimes C. \]

Now 
\[ v(\triangle BDE \otimes C) = v(\triangle ABD \otimes C) \] (1)

by Theorem 1, since 
\[ \triangle BDE \otimes C \cup \triangle ABD \otimes C = \square ABED \otimes C \]

and \( BD \) is a diagonal of the parallelogram \( \square ABED \).

Moreover, since 
\[ \triangle BDE \otimes C \equiv \triangle BCE \otimes D \quad \text{and} \quad \triangle DEF \otimes C \equiv \triangle CEF \otimes D, \]

it is apparent that their union is also a parallelogram-based pyramid. That is, 
\[ \triangle BCE \otimes D \cup \triangle CEF \otimes D = \square BCFE \otimes D \]

Thus, the parallelogram-based pyramid is formed by merging the two triangular – based pyramids along their common side \( CE \).

So it also follows from Theorem 1 that 
\[ v(\triangle BCE \otimes D) = v(\triangle CEF \otimes D). \]

That is, 
\[ v(\triangle BDE \otimes C) = v(\triangle DEF \otimes C) \] (2)

Now combining (1) and (2), we obtain the desired result. Namely, that each of the three triangular pyramids in the partition of the triangular prism have equal volume. Hence each has a volume equal to one-third that of the containing prism.
**Theorem 3:** The formula for the volume of a triangular based pyramid is one third that of the area of the base triangle times the height of the pyramid.

Proof:
Since the volume of any triangular prism is that of a cylinder having a triangular base, we know that

\[ V_{PRISM} = \text{area (base } \triangle \text{)} \cdot \text{height} \]

It follows from Theorem 2 that

\[ V_{\triangle-BASED \ PYRAMID} = \frac{1}{3} \text{area (base } \triangle \text{)} \cdot \text{height} \]

**Theorem 4:** The volume of a right circular cone with base radius \(r\) and height \(h\) is

\[ V_{CONE} = \frac{1}{3} \pi r^2 \cdot h \]

Proof:
We may apply the method of exhaustion to find the volume of the circular cone in much the same way as we used it earlier to prove the area formula for the circle. We begin by inscribing a regular \(2^n\)–gon on the base circle for \(n = 2, 3, 4 \ldots\).

Let \(\triangle OP_i P_{(i+1)}\) be one of the \(2^n\) congruent isosceles triangles in the partition of the \(2^n\)–gon by joining an adjacent pair of its vertices to the center, \(O\), and suppose that the vertex of the cone is the point \(V\). Then the volume, \(V_{n, i}\), of the pyramid \(\triangle OP_i P_{(i+1)} \oplus V\) is given by

\[ V_{n, i} = \frac{1}{3} \text{Area}(\triangle OP_i P_{(i+1)}) \cdot OV, \]

and the volume of the cone is approximated by the sum of the volumes of the \(2^n\) inscribed right triangular pyramids:
\[
V_{\text{CONE}} \approx V_{n_1} + V_{n_2} + V_{n_3} + \ldots + V_{n_n}
\]
\[
= \frac{1}{3} \text{Area}(\triangle OP_1P_2) \cdot OV + \frac{1}{3} \text{Area}(\triangle OP_2P_3) \cdot OV + \frac{1}{3} \text{Area}(\triangle OP_3P_4) \cdot OV
\]
\[\ldots + \frac{1}{3} \text{Area}(\triangle OP_{n-1}P_1)\]
\[
= \frac{1}{3} \left[ \text{Area}(\triangle OP_1P_2) + \text{Area}(\triangle OP_2P_3) + \text{Area}(\triangle OP_3P_4) + \ldots + \text{Area}(\triangle OP_{n-1}P_1) \right] \cdot OV
\]

Then as \( n \to \infty \), the difference in volume between that of the pyramid with the \( 2^n \)-gon for a base and the right circular cone will become arbitrarily small. That is,

\[
V_{\text{CONE}} = \lim_{n \to \infty} \left[ V_{n_1} + V_{n_2} + V_{n_3} + \ldots + V_{n_n} \right]
\]
\[
= \lim_{n \to \infty} \frac{1}{3} \left[ \text{Area}(\triangle OP_1P_2) + \text{Area}(\triangle OP_2P_3) + \text{Area}(\triangle OP_3P_4) + \ldots + \text{Area}(\triangle OP_{n-1}P_1) \right] \cdot OV
\]
\[
= \frac{1}{3} \pi r^2 \cdot OV
\]

Therefore,

\[
V_{\text{CONE}} = \frac{1}{3} (\pi r^2) \cdot (\text{height}).
\]

**Theorem 5:** The volume of any circular cone with base radius \( r \) and height \( h \) is

\[
V_{\text{CONE}} = \frac{1}{3} \pi r^2 \cdot h
\]

**Proof:**
This follows from the fact that any two circular cones with equal bases and the same height have identical cross sections in the planes parallel to their bases.

Let two cones: \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) share the same circular base having center \( O \).

Let \( \mathcal{C}_1 \) have the vertex \( V_1 \) and be such that \( V_1O \) is perpendicular to the base circle and let \( \mathcal{C}_2 \) have its vertex at \( V_2 \) and be such that the distance from \( V_2 \) to the base circle is equal to \( \|V_1O\| = H \).

Now there must be a plane \( \mathcal{P} \) through the vertices \( V_1 \) and \( V_2 \) that is perpendicular to the base circle and which also intersects the base circle in a diameter \( AB \). Consequently, \( \mathcal{P} \) will intersect \( \mathcal{C}_1 \) in the triangle \( \triangle ABV_1 \) and \( \mathcal{P} \) will intersect \( \mathcal{C}_2 \) in the triangle \( \triangle ABV_2 \).

Now take any other plane \( \mathcal{Q} \) parallel to the base plane. This plane \( \mathcal{Q} \) will intersect \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) in circles having the diameters \( CD \) and \( EF \) respectively.
That is,
\[ P \cap \emptyset \cap C_1 = CD \quad \text{and} \quad P \cap \emptyset \cap C_2 = EF. \]

Now because \( \triangle ABV_1 \sim \triangle CDV_1 \) we have that
\[ \frac{CD}{AB} = \frac{h_1}{H} \quad (1) \]
where \( h_1 = \text{altitude}(\triangle BCV_1) \) from vertex \( V_1 \)

and because \( \triangle ABV_2 \sim \triangle EFV_2 \) we have that
\[ \frac{EF}{AB} = \frac{h_2}{H} \quad (2) \]
where \( h_2 = \text{altitude}(\triangle BCV_2) \) from vertex \( V_2 \).

Since \( P \perp \emptyset \) we know that \( h_1 = h_2 \). So from (1) and (2) it follows that
\[ \frac{CD}{AB} = \frac{EF}{AB} \]
\[ \therefore CD = EF. \]

So the cross sections of both of the cones have equal diameters and hence, equal areas. Therefore, the area of any circular cone is
\[ V_{CONE} = \frac{1}{3} \pi r^2 \cdot h. \]

*To establish equality (2) in the case of the figure to the right, observe that
\[ \frac{EG}{AI} = \frac{h_2}{H} \quad \text{since} \quad \triangle EGV_2 \sim \triangle AV_2 \quad \text{and} \quad \frac{h_2}{H} = \frac{FG}{BI} \quad \text{since} \quad \triangle BIV_2 \sim \triangle FGV_2. \]

So
\[ \frac{EG}{AI} = \frac{FG}{BI}. \]

That is,
\[ \frac{EF + FG}{AB + BI} = \frac{FG}{BI}. \]

Cross multiply to obtain:
\[ BI \cdot EF + BI \cdot FG = FG \cdot AB + FG \cdot BI \]
\[ \Rightarrow BI \cdot EF = FG \cdot AB \]
\[ \therefore \frac{FG}{BI} = \frac{EF}{AB}. \]

but recall that
\[ \frac{FG}{BI} = \frac{h_2}{H} \quad \text{(since} \quad \triangle FGV_2 \sim \triangle BIV_2). \]

Now it follows immediately from these above two equalities that
\[ \frac{EF}{AB} = \frac{h_2}{H} \] as required.
**Theorem 6:** The volume of a sphere of radius $r$ is given by the formula:

$$V_{SPHERE} = \frac{4}{3} \pi r^3$$

**Proof:**

We first establish the equality of the volume of the bowl shaped region bounded beneath a cone inscribed in a circular cylinder having height $r$ and base radius $r$ with that of the hemisphere of radius $r$.

For this, it suffices to verify that the area of the ring, $A_y$, is the same as that of the disc $a_y$, for all heights $0 < y < r$ in the figure below.

In the cylinder, observe that $\frac{r-y}{l} = \frac{r}{r}$ by triangle similarity.

Thus, $r-y = l \implies y = r-l$.

So the area of the ring, $A_y$, is given by:

$$A_y = \pi r^2 - \pi (r-l)^2 = \pi [ r^2 - y^2 ].$$

Evidently, the radius of the disc, $a_y$, is

$$a_y = \pi R^2 \text{ where } R = \sqrt{r^2 - y^2} \text{ due to the Pythagorean Theorem.}$$

Hence,

$$a_y = \pi [ r^2 - y^2 ] = A_y \text{ for all } 0 < y < r.$$  

It now follows from Cavalieri’s Principle that the volume, $V_{BOWL}$, of the bowl shaped region beneath the cone is precisely the same as that of the hemisphere. So

$$V_{BOWL} = V_{CYLINDER} - V_{CONE}$$

$$= ( \pi r^2 ) \cdot r - \frac{1}{3} ( \pi r^2 ) \cdot r$$

$$= \pi r^3 - \frac{1}{3} \pi r^3$$

$$= \frac{2}{3} \pi r^3 = V_{HEMISPHERE}$$

$$\therefore \ V_{SPHERE} = 2 \cdot V_{HEMISPHERE} = \frac{4}{3} \pi r^3.$$

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