Nonrepetitive colorings of graphs of bounded tree-width

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January 22, 2006

Abstract

A sequence of the form $s_1 s_2 \dots s_m s_1 s_2 \dots s_m$ is called a *repetition.* A vertex-coloring of a graph is called *nonrepetitive* if none of its paths is repetitively colored. We answer a question of Grytczuk [5] by proving that every outerplanar graph has a nonrepetitive 12-coloring. We also show that graphs of tree-width t have nonrepetitive 4^t-colorings.

1 Introduction

A *block* of a sequence is a subsequence consisting of consecutive terms of the sequence. A sequence of the form $s_1 s_2 \dots s_m s_1 s_2 \dots s_m$ is called a *repetition.* A sequence is called *nonrepetitive* if none of its blocks is a repetition. More than 100 years ago Thue [9] proved that there are nonrepetitive sequences of arbitrary length using only the symbols $1, 2,$ and 3. This result has been extended in many directions and a recent survey of Grytczuk [5] collects a substantial number of such results. In this paper we consider an extension to graph theory suggested by Alon, Grytczuk, Hałuszczak, and Riordan [1].

A *walk* in a graph is a sequence of vertices such that consecutive vertices are joined by an edge. A *path* is a walk in which all vertices are distinct. For any set of colors $\mathbf{C}$, a *coloring* of a graph G is a function $c : V(G) \to \mathbf{C}$. If c uses at most k colors, then c is called a *k-coloring.* A *pattern* is a sequence of (not necessarily distinct) colors. If B is a collection of "bad" patterns, then we say that a coloring of G is *B-free* if for every path $v_1 v_2 \dots v_k$ we have $c(v_1)c(v_2)\dots c(v_k) \notin B$. Let B_m be the collection of all repetitions of length at most

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$2m$ over $\mathbf{C}$. For example, $B_1 = \{xx : x \in \mathbf{C}\}$ and $B_2 = B_1 \cup \{xyxy : x, y \in \mathbf{C}\}$. Thus a B_1-free coloring is just a proper vertex-coloring in the usual sense: adjacent vertices receive different colors. A B_2-free coloring is a proper coloring in which no path on 4 vertices is 2-colored, or equivalently any two color classes induce a star forest. Consequently, B_2-free colorings have been studied under the name *star coloring* (see [2, 4, 6]).

Here we are particularly interested in *nonrepetitive colorings*, that is, B_∞-free colorings, where B_∞ denotes the set of all repetitions over $\mathbf{C}$. Our investigation is motivated by the following question of Alon, Grytczuk, Hałuszczak, and Riordan [1].

Question 1 *Is there a constant k such that every planar graph has a nonrepetitive k-coloring?*

In [2] a planar graph is given which has no star coloring with 9 colors, so k would have to be at least 10. The answer to Question 1 is still open, but we provide some evidence suggesting an affirmative answer. At the Budapest workshop in honor of Miklós Simonovits' 60th birthday, Grytczuk suggested replacing planar by outerplanar as a first point of attack (see also [5]).

Question 2 *Is there a constant k such that every outerplanar graph has a nonrepetitive k-coloring?*

In Sections 2 and 3 we collect several technical lemmas. In Section 4 we answer Question 2 in the affirmative by showing that every outerplanar graph has a nonrepetitive 12-coloring. In a similar fashion we show that graphs of tree-width at most t have nonrepetitive 4^t-colorings. Then using a theorem of Robertson and Seymour (see Section 5) it follows that planar graphs form the smallest family of graphs closed under taking minors for which it is unclear if its members can be nonrepetitively colored with a bounded number of colors.

Finally, in Section 6 we construct 2-degenerate graphs of girth $2m+2$ which have B_m-free 5-colorings but no B_{m+1}-free k-colorings. This construction shows that there are 2-degenerate bipartite graphs of arbitrarily large girth, which require arbitrarily many colors for a B_∞-free coloring. Thus several natural approaches to Question 1 are unlikely to be successful.

2 A palindrome lemma

A sequence $S = s_1 s_2 \ldots s_k$ is called a *palindrome* if $S = s_k s_{k-1} \ldots s_1$ and $k \geq 2$. If no block of a sequence is a palindrome, then we call it *palindrome-free*. Since every palindrome contains a palindrome of the form xx or xyx in the middle, it follows that it is enough to avoid palindromes of length 2 and 3.

As mentioned in [1] it is easy to see that there are arbitrarily long sequences on 4 symbols which are nonrepetitive and palindrome-free: starting from a long nonrepetitive sequence on $\{1, 2, 3\}$, insert the symbol 4 between consecutive blocks of length two. For example, from the sequence 123132123 we obtain 1243143241243. Since a palindrome-free sequence of length 6 on 3 symbols is a repetition, it follows that in general 4 symbols are necessary.

The following lemma will be crucial.

Lemma 3 *If c is a nonrepetitive palindrome-free coloring of a path P, and P' is obtained from P by adding a loop at each vertex, then every repetitively colored walk W_1W_2 in P' satisfies $W_1 = W_2$.*

Proof. Let $W_1 = v_1v_2 \dots v_m$, and $W_2 = v_{m+1}v_{m+2} \dots v_{2m}$. Since c is palindrome-free, $v_i = v_{i+1}$ if and only if $c(v_i) = c(v_{i+1})$ and $v_i = v_{i+2}$ if and only if $c(v_i) = c(v_{i+2})$. We proceed by induction on m. For $m = 1, 2$ the result is easy to check using the above observations.

Let $m \geq 3$. Suppose first that $v_i = v_{i+1}$ for some $i < m$. Since W_1W_2 is repetitively colored it follows that $c(v_{m+i}) = c(v_i) = c(v_{i+1}) = c(v_{m+i+1})$, so that $v_{m+i} = v_{m+i+1}$. Thus we can omit v_i from W_1 and v_{m+i} from W_2 and obtain a repetitively colored walk of length $2m - 2$. The result follows by induction. We may now assume that $v_i \neq v_{i+1}$ for all $i < m$, and by similar reasoning, for $m \leq i < 2m$.

Suppose next that for some $i \leq m - 2$ we have $v_i = v_{i+2}$. Since $c(v_{m+i+2}) = c(v_{i+2}) = c(v_i) = c(v_{m+i})$, it follows that $v_{m+i} = v_{m+i+2}$. Thus by letting $W_1' = v_1v_2 \dots v_iv_{i+3} \dots v_m$ and $W_2' = v_{m+1}v_{m+2} \dots v_{m+i}v_{m+i+3} \dots v_{2m}$, we obtain a shorter repetitively colored walk $W_1'W_2'$. By induction, $W_1' = W_2'$. Now it suffices to show that $v_{i+1} = v_{m+i+1}$. Since $v_i = v_{m+i}$, $v_i \neq v_{i+1}$, and $v_{m+i} \neq v_{m+i+1}$, $v_{i+1}v_iv_{m+i+1}$ is a walk in P. Since $c(v_{i+1}) = c(v_{m+i+1})$, $c(v_{i+1})c(v_i)c(v_{m+i+1})$ is a palindrome. Then $v_{i+1}v_iv_{m+i+1}$ is not a path, so $v_{i+1} = v_{m+i+1}$.

If we are in neither of the two previous cases, then W_1 and (by similar reasoning) W_2 are subpaths of P. We immediately obtain a contradiction to the fact that c is nonrepetitive and palindrome-free: If W_1 and W_2 go in the same direction along P, then depending on whether v_{m+1} equals v_{m-1}, v_m, or neither, either $(W_1 - v_{m-1} - v_m)(W_2 - v_{2m-1} - v_{2m})$, $(W_1 - v_m)(W_2 - v_{2m})$, or W_1W_2 is a repetitively colored subpath of P. If W_1 and W_2 are paths in the opposite direction along P, then depending on whether v_{m+1} equals v_{m-1}, v_m, or neither, either $W_1 - v_m$, W_1, or $W_1 - v_1$ is a palindrome. ■

3 Level partitions

We now translate Lemma 3 into the form in which we will apply it. Suppose that $V(G)$ is partitioned into *levels* $V_1, V_2, \dots, V_m$ such that the neighbors of every vertex in V_i are in $V_{i-1} \cup V_i \cup V_{i+1}$. For a given partition $V_1, V_2, \dots, V_m$ and a sequence $S = s_1 \dots s_m$ let c_S denote the coloring of G given by $c_S(v) = s_i$ for all $v \in V_i$. The *level pattern* of a path $v_1 \dots v_r$ is the sequence of levels $k_1 \dots k_r$ such that $v_i \in V_{k_i}$.

Lemma 4 *Suppose that $V(G)$ is partitioned into levels as above. If S is a palindrome-free nonrepetitive sequence and P_1P_2 is a repetitively colored path under c_S, then P_1 and P_2 have the same level pattern.*

Proof. Let P be the path on vertices $1, \dots, m$, and let P' be the multigraph obtained from P by adding a loop at each vertex. Observe that the level pattern of P_1P_2 is a walk W_1W_2 in P'. Color P' with $c(i) = s_i$. Since P_1P_2 is repetitively colored under c_S, it follows that W_1W_2 is a repetitively colored walk in P'. Thus by Lemma 3, $W_1 = W_2$ as desired. ■

Definition 5 Let $V_1, \dots V_m$ be a partition of $V(G)$ and let G_k, $G_{<k}$, and $G_{>k}$ denote the subgraphs of G induced by V_k, $V_1 \cup V_2 \cup \dots \cup V_{k-1}$, and $V_{k+1} \cup V_{k+2} \cup \dots \cup V_m$, respectively. The *$k$-shadow* of a subgraph H of G is the set of vertices in V_k which have a neighbor in $V(H)$. We say that G is *shadow complete* (with respect to the partition) if the k-shadow of every component of $G_{>k}$ induces a complete graph.

We now use Lemma 4 to prove a theorem which easily implies our main results.

Theorem 6 *If G is shadow complete and each G_k has a nonrepetitive coloring with b colors, then G has a nonrepetitive coloring with $4b$ colors.*

Proof. Let $V(G)$ be decomposed into the levels $V_1, V_2, \dots V_m$ and let h_k be a nonrepetitive coloring of G_k using colors from $\{1, 2, \dots, b\}$. Furthermore, let $S = s_1s_2 \dots s_m$ be a nonrepetitive palindrome-free sequence on at most 4 symbols. Consider the coloring which colors a vertex $u \in V_k$ with the color $c(u) = (s_k, h_k(u))$. This coloring c uses at most $4b$ colors, and it remains to check that it is nonrepetitive.

Suppose that $P = P_1P_2$ is repetitively colored under c. Since the first coordinate of $c(u)$ is $c_S(u)$ it follows that P is repetitively colored under c_S. Hence, by Lemma 4, P_1 and P_2 have the same level pattern.

Let k be the smallest integer such that V_k intersects P and consider the subsequence of vertices in P that are in V_k. Since P_1 and P_2 have the same level pattern it follows that this subsequence is repetitively colored with respect to h_k. We obtain a contradiction by observing that this sequence is a path in G_k: If u immediately precedes v in the subsequence, then either they are consecutive in P (and thus adjacent) or they are connected by a path in $G_{>k}$, in which case they are adjacent since G is shadow complete. ■

4 A level partition for chordal graphs

The *k-th iterated neighborhood* of a fixed vertex $x \in V(G)$ is the set of vertices at distance k from x, $N^k(x) = \{v \in V(G)\colon d(v,x) = k\}$. If G is a connected graph, then the iterated neighborhoods partition $V(G)$. We now show that if G is a chordal graph, then this partition is also shadow complete. Recall that a graph is *chordal* if it contains no induced cycle on more than 3 vertices, or equivalently, if there is a *simplicial construction ordering* of its vertices $v_1, \ldots, v_n$ such that for each v_i the neighbors preceding it induce a complete subgraph of G.

Lemma 7 *Let G be a connected chordal graph with clique number $\omega > 1$ and let x be any vertex. G is shadow complete with respect to the partition given by $V_k = N^k(x)$ and every G_k is a chordal graph with clique number $< \omega$.*

Proof. Let x be an arbitrary vertex of G, and let H be a component of $G_{>k}$. Observe that $N^0(x) = \{x\}$, so that it suffices to consider $k > 0$. In this case the k-shadow of H is a minimal set of vertices separating x from H, so that by a result of Dirac [10, Exercise 5.3.27 b)] it must induce a complete graph.

Every induced subgraph of a chordal graph is chordal, so G_k is chordal. To see that G_k has clique number $< \omega$, consider a simplicial construction ordering consisting of the vertices in $N^0(x), N^1(x), N^2(x), \ldots$ in this order (this can be found by [10, Lemma 5.3.16]). For any clique K in G_k, let v be the last vertex of K in this ordering. Observe that v has a neighbor u in $N^{k-1}(x)$, and by the properties of the ordering $K + u$ is a complete graph so that $|K + u| \leq \omega$ as desired. ■

As an important special case we obtain the following result for outerplanar graphs.

Lemma 8 *If G is maximal outerplanar, then G is shadow complete with respect to the partition given by $V_k = N^k(x)$ and every G_k is a linear forest.*

Proof. Since maximal outerplanar graphs are chordal graphs the shadow completeness of G is immediate. Furthermore, G has clique number at most 3 since K_4 is not outerplanar, so that every G_k is a chordal graph with clique number at most 2, i.e. a forest. To see that every component C of G_k is a path, contract all of $G_{<k}$ to x and observe that every vertex in C is now adjacent to x. Thus if C would contain a vertex y and three of its neighbors v_1, v_2, v_3, then $\{x, y, v_1, v_2, v_3\}$ would yield a $K_{2,3}$-minor of G, a contradiction to the fact that $K_{2,3}$ is not outerplanar. ■

Since by Thue's original result, linear forests have nonrepetitive 3-colorings, Lemma 8 and Theorem 6 immediately imply the following result.

Corollary 9 *Every outerplanar graph has a nonrepetitive* 12*-coloring.*

In [2, 4] it is shown that outerplanar graphs can be star colored with 6 colors, and that this is best possible. Since nonrepetitive colorings are star colorings it follows that 12 can not be replaced by 5 in the previous corollary, although 12 is probably not best possible.

Basically the same proof also yields a result for graphs with bounded tree-width. See the book of Diestel [3] for an introduction to tree-width.

Corollary 10 *Every graph of tree-width* $t \geq 0$ *has a nonrepetitive* 4^t*-coloring.*

Proof. By Proposition 12.3.12 of [3], every graph of tree-width t is a subgraph of a chordal graph with clique number $t + 1$, so it suffices to prove by induction that chordal graphs of clique number $t + 1$ have nonrepetitive 4^t-colorings. For the base step $t = 0$ observe that chordal graphs with clique number 1 are edgeless and thus have nonrepetitive 1-colorings. The inductive step follows by combining Lemma 7 and Theorem 6. ■

Observe that for $t = 0, 1$ this result is best possible. In [2] it is shown that every graph of tree-width t has an star coloring with $\binom{t+2}{2}$ colors and that this is best possible, so 4^t cannot be replaced by $\binom{t+2}{2} - 1$ in Corollary 10. There is a big gap between $\binom{t+2}{2}$ and 4^t, and it remains to be seen if there is a polynomial upper bound.

5 Nonrepetitive colorings of planar graphs

Corollary 10 has several implications for nonrepetitive colorings of planar graphs. Let X be any graph and $\mathcal{F}$ be the family of all graphs not containing X as a minor. Robertson and Seymour [7, 8] proved that the graphs in $\mathcal{F}$ have bounded tree-width if and only if X

is planar. Thus when X is planar it follows that the graphs in $\mathcal{F}$ can be nonrepetitively k-colored for some k which depends only on X.

Let $\mathcal{F}$ be a family of graphs closed under taking minors, such that its members cannot be nonrepetitively colored with a bounded number of colors. By the above observation $\mathcal{F}$ must include all planar graphs. Thus Question 1 simply asks if it is possible that $\mathcal{F}$ consists entirely of planar graphs. This shows that Question 1 is the "right question" to study for gaining a deeper understanding of the nature of nonrepetitive graph colorings.

Corollary 10 implies that planar graphs which do not have nonrepetitive colorings with few colors must have large tree-width. The standard example of graphs with large tree-width are the *grid* graphs, $P_m \square P_m$. The vertices of $P_m \square P_m$ are pairs (x, y) with $x, y \in \{1, 2 \dots m\}$ and two such vertices are adjacent if they agree in one entry and differ by one in the other. The following result is probably far from optimal, but at least provides some evidence that planar graphs have nonrepetitive colorings with a bounded number of colors.

Theorem 11 *Every grid graph admits a nonrepetitive 16-coloring.*

Proof. Let $S = s_1 s_2 \dots s_m$ be a nonrepetitive palindrome-free sequence on $\{1, 2, 3, 4\}$. Consider the coloring given by $c((x, y)) = (s_x, s_y)$. This coloring uses at most 16 colors and it suffices to show that it is nonrepetitive. Suppose that there is a path $P_1 P_2$ such that the paths P_1 and P_2 have identical colors. Let the first vertex of P_i be $u_i = (x_i, y_i)$. Since $u_1 \neq u_2$ we can assume by symmetry that $x_1 \neq x_2$. Using the levels $V_k = \{(k, y) : 1 \leq y \leq m\}$ it follows that $P_1 P_2$ is repetitively colored under c_S. Thus by Lemma 4, P_1 and P_2 have the same level pattern, so that $x_1 = x_2$, a contradiction. ■

Note that virtually the same proof as in Theorem 11 can be used to show the stronger result that the strong product of t paths admits a nonrepetitive coloring with at most 4^t colors. Here the strong product of G and H is the graph with vertex set $V(G) \times V(H)$ in which distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent when $d_G(x_1, x_2) \leq 1$ and $d_H(y_1, y_2) \leq 1$.

6 A construction

Let $K(n, n)$ denote the complete bipartite graph with parts $X = \{x_1, \dots x_n\}$ and $Y = \{y_1, \dots y_n\}$.

Lemma 12 *Let $n = t! + 1$ for $t \geq 1$. For every t-coloring of the edges of $K(n, n)$ there are distinct edges $x_i y_j$ and $x_j y_k$ which have the same color.*

Proof. We proceed by induction on t. For $t = 1$ the edges x_1y_2 and x_2y_2 suffice. For $t > 1$ observe that every vertex x_j must be missing some color c_j, since otherwise picking an arbitrary edge x_iy_j (with $i \neq j$) we can find some edge x_jy_k with the same color. By the pigeonhole principle some color is missing on at least $n' = (t-1)! + 1$ vertices, say $x_1, \dots x_{n'}$. Thus we have a $(t-1)$-coloring of the edges of $K(n', n')$ and the result follows by induction. ■

Theorem 13 *For all positive integers m, t, there is a graph G of girth $2m + 2$ which has a B_m-free 5-coloring but no B_{m+1}-free t-coloring.*

Proof. Let m, t be given and let $N = (t^{m+1} + 1)! + 1$. Consider $K = K(N, N)$ and let M be the perfect matching $x_1y_1, \dots x_Ny_N$. Let G be obtained from K by subdividing each edge $x_iy_j \notin M$ exactly $m-1$ times, thus obtaining a path $P_{i,j}$ on $m+1$ vertices. Since every cycle in K has length at least 4 and contains at least 2 edges not in M the girth of G must be (at least) $4 + 2(m-1)$. (Observe that G is bipartite when m is even, and 3-chromatic otherwise.)

To obtain a B_m-free 5-coloring of G color the vertices in X blue, in Y red, and color the subdivision vertices with 3 new colors so that every subdivision path is nonrepetitively colored. Since $G - X$ forms a star forest of diameter $2m - 2$ every path on $2m$ vertices must contain a blue vertex. Thus every repetitively colored path P on $2m$ vertices contains 2 blue vertices u, v at distance at most m. But since M contains no u, v-path, every such path in G must consist of at least $m + 2$ vertices, a contradiction.

Now consider a t-coloring of G. Associate with this coloring a coloring of the edges of $K(N, N)$ where $c(x_iy_i) = 0$ and $c(x_iy_j)$ (for $i \neq j$) is the color pattern of $P_{i,j}$. This associated coloring uses at most $1 + t^{m+1}$ colors, so by Lemma 12 we have $c(x_iy_j) = c = c(x_jy_k)$ for $x_iy_j \neq x_jy_k$. Since color 0 forms a matching it now follows that $c \neq 0$ and thus $j \neq i, k$. Hence $P_{i,j}P_{j,k}$ is a repetitively colored path on $2(m + 1)$ vertices. ■

No effort has been made to minimize the number of vertices in the graph. The number 5 is probably not best possible but it can't be replaced by 2 (for $m > 1$) since such a graph would need to be star colorable with 2 colors, and thus a star forest. A simple way to reduce 5 to 4 in general would be if there were repetition-free palindromes on 3 symbols of arbitrary length, since this could be used to color the star forest $G - X$. Of course when $m = 1$, then G has a proper 2-coloring and when $m = 2$, then it has a star coloring with 3 colors, but for large m, 4 colors seems to be the best we can hope for.

This simple construction has many nice properties: It is 2-degenerate (since $X \cup Y$ induces a matching, M, in G) and its average degree is $2+\epsilon$, so that degeneracy arguments are unlikely to solve Question 1. It also shows that there are bipartite graphs with arbitrarily high girth and Thue number.

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