Rational Implementation of Distributed Delay Using Extended Bilinear Transformations

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Outline

- Introduction
- Problem formulation
- Rational Implementation
- Stability and convergence of the implementation
- Numerical example
What is distributed delay?

A finite integral over the time:

\[(1) \quad v(t) = \int_0^h e^{A\zeta} Bu(t - \zeta) d\zeta, \quad (h > 0).\]

The equivalent in the \(s\)-domain:

\[(2) \quad Z(s) = (I - e^{-(sI-A)h})(sI - A)^{-1}B.\]

It often appears in the controller of a delay system as

- part of finite-spectrum-assignment control law
- modified Smith predictor

The problem is how to implement \(Z\) to guarantee that \(Z\) is stable even if \(A\) is unstable?
Implementation in the $\mathcal{Z}$-domain

(a) Corresponding to $Z_f(s) = \frac{1-e^{-s\frac{h}{N}}}{s} \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B$

(b) Cor. to $Z_{f0}(s) = \frac{1-e^{-\frac{h}{N}s}}{s} \frac{e^{\frac{h}{N}A-I}}{\frac{h}{N}} A^{-1} \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B$
Implementation in the $s$-domain

\[
Z_{f\epsilon}(s) = \frac{1 - e^{-\frac{h}{N}(s+\epsilon)}}{1 - e^{-\frac{h}{N}\epsilon}} \frac{e^{\frac{h}{N}A}}{s/\epsilon + 1} A^{-1} \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B.
\]

\[
\lim_{N \to +\infty} \|Z_{f\epsilon}(s) - Z(s)\|_{\infty} = 0, \quad (\epsilon \geq 0).
\]

- an extra parameter $\epsilon$
- involving delay terms
- rational implementation?
Rational implementation

\[ \Pi = (sI - A + \Phi)^{-1} \Phi, \]
\[ \Phi = (\int_0^N e^{-A\zeta} d\zeta)^{-1}. \]
slowly convergent
Keep points about implementations

- The low-pass property of distributed delay must be kept in the implementation;

- There is no unstable pole-zero cancellation in the implementation in order to guarantee the internal stability;

- The implementation itself must be stable to guarantee the stability of the closed-loop system;

- The implementation error should be able to be made small enough to guarantee the stability of the closed-loop system;

- The static gain of the distributed delay should be retained in the implementation to guarantee the steady-state performance of the system.
Problem formulation

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Problem formulation

How to find an approximation, which is

- stable
- fast-convergent
- and rational

for the following finite-impulse-response block to meet the requirements on the previous slide?

\[ Z(s) = (I - e^{-(sI-A)h})(sI - A)^{-1}B. \]
Rational Implementation

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The $\gamma$-operator

The well-known $\gamma$-operator in digital and sampled-data control circles is defined as

$$\gamma = \frac{2}{\tau} \cdot \frac{q - 1}{q + 1},$$

where $q$ is the shift operator and $\tau$ the sampling period.

- often used to digitizing a continuous-time transfer function
- also called the bilinear transformation, or the Tustin’s transformation
- corresponds to the trapezoidal rule
- also connects to the (lower) linear fractional transformation $F_l$ and the (right) homographic transformation $H_r$:

$$\gamma = \frac{2}{\tau} F_l\left( \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, q \right) = \frac{2}{\tau} H_r\left( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, q \right).$$
Properties of the $\gamma$-operator

$$q = \mathcal{H}_r \left( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}, \frac{\tau}{2} \gamma \right) = \frac{1 + \frac{\tau}{2} \gamma}{1 - \frac{\tau}{2} \gamma}.$$ 

Since $q \to e^{\tau s}$ when $\tau \to 0$,

$$e^{-\tau s} \approx q^{-1} = \frac{1 - \frac{\tau}{2} \gamma}{1 + \frac{\tau}{2} \gamma}.$$ 

Furthermore, $\gamma$ holds the following limiting property:

$$\lim_{\tau \to 0} \gamma = \lim_{\tau \to 0} \frac{2 e^{\tau s} - 1}{\tau e^{\tau s} + 1} = s.$$ 

$\gamma$ can be regarded as an approximation of the differential operator $p = \frac{d}{dt}$. 
Extended bilinear transformation $\Gamma$

The $\gamma$-operator:

$$\gamma = \frac{2}{\tau} \cdot \frac{q - 1}{q + 1}$$

Define

$$(3) \quad \Gamma = (e^{\tau(sI-A)} - I)(e^{\tau(sI-A)} + I)^{-1}\Phi,$$

with $\tau = \frac{h}{N}$ and

$$\Phi = \left( \int_0^{\frac{h}{N}} e^{-A\zeta} \, d\zeta \right)^{-1}(e^{-A\frac{h}{N}} + I).$$
Properties of $\Gamma$

(i) the limiting property

$$sI - A = \lim_{\tau \to 0} \Gamma,$$

(ii) the static property

$$sI - A|_{s=0} = \Gamma|_{s=0} = -A,$$

(iii) the cancellation property

$$\Gamma|_{sI - A=0} = 0.$$

Due to the above properties, this $\Gamma$ is able to bring a rational implementation to guarantee the stability of the closed-loop system and the steady-state performance.
Approximation of $Z$

From (3), we have

$$e^{-(sI-A)rac{h}{N}} = (\Phi - \Gamma)(\Phi + \Gamma)^{-1}.$$ 

Substitute this into $Z$, then

$$(5) \quad Z(s) = (I - (\Phi - \Gamma)^N(\Phi + \Gamma)^{-N})(sI - A)^{-1}B.$$ 

Since $\Gamma \approx sI - A$, $Z$ can be approximated as

$$(6) \quad Z_r(s) = \left(I - (\Phi - sI + A)^N(sI - A + \Phi)^{-N}\right)(sI - A)^{-1}B$$

$$= \left(I - (\Phi - sI + A)(sI - A + \Phi)^{-1}\right)$$

$$\cdot \sum_{k=0}^{N-1} (\Phi - sI + A)^k(sI - A + \Phi)^{-k}(sI - A)^{-1}B$$

$$= 2(sI - A + \Phi)^{-1}\sum_{k=0}^{N-1} (\Phi - sI + A)^k(sI - A + \Phi)^{-k}B$$

$$(7) \quad = \sum_{k=0}^{N-1} \Pi^k \Xi B,$$

with

$$\Pi = (\Phi - sI + A)(sI - A + \Phi)^{-1}, \quad \Xi = 2(sI - A + \Phi)^{-1}.$$
Structure of $Z_r$

\[
Z_r = \sum_{k=0}^{N-1} \Pi^k \Xi B
\]

- bi-proper nodes $\Pi$ plus a strictly proper node $\Xi$
- no zero-pole cancellation
- if all the nodes are stable then $Z_r$ is stable
- $Z_r$ converges to $Z$ when $N \to +\infty$

\[
\Pi = (sI - A + \Phi)^{-1} (A - sI + \Phi) \\
\Xi = 2(sI - A + \Phi)^{-1}
\]
Stability and convergence

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Stability of the nodes

Denote an eigenvalue of $A$ as $\bar{\sigma} + j\bar{\omega}$. Then the corresponding eigenvalue of $\frac{h}{N}A$ is $\sigma + j\omega$ with $\sigma = \frac{h}{N}\bar{\sigma}$ and $\omega = \frac{h}{N}\bar{\omega}$.

**Theorem 1** The following conditions are equivalent:

- $Z_r$, $\Pi$, $\Xi$ or $A - \Phi$ is stable;
- $\int_0^{\frac{h}{N}} e^{A\zeta} d\zeta$ is antistable;
- $\sigma \cos \omega + \omega \sin \omega - \sigma e^{-\sigma} > 0$, ignoring the case when $\sigma = 0$ and $\omega = 0$.

**Note:** If all the eigenvalues of $A$ are real, then each node $\Pi$ or $\Xi$ is stable for any natural number $N$. 
\[ f(\sigma, \omega) = \sigma \cos \omega + \omega \sin \omega - \sigma e^{-\sigma} \]
A sufficient condition for the stability of nodes

The conditions in Theorem 1 are all satisfied, i.e., the nodes are stable, for any number $N > \underline{N}$ with

$$N = \left\lceil \frac{h}{2.8} \cdot \max_i |\lambda_i(A)| \right\rceil,$$

where $\lceil \cdot \rceil$ is the ceiling function.
Convergence of the implementation

\[
\Phi \frac{h}{N} = \frac{h}{N} \left( \int_0^h e^{-A\zeta} d\zeta \right)^{-1} \left( e^{-A\frac{h}{N}} + I \right)
= A \frac{h}{N} \left( I - e^{-A\frac{h}{N}} \right)^{-1} \left( e^{-A\frac{h}{N}} + I \right)
\]

is antistable if \( N > N \).
Theorem 2 Denote the approximation error of \( Z_r \) as \( E_r = Z - Z_r \). Then

\[
\lim_{N \to +\infty} \| E_r(s) \|_{\infty} = 0.
\]

This theorem indicates that there always exists a number \( N \) such that the implementation is stable and, furthermore, the \( H^\infty \)-norm of the implementation error is less than a given positive value. According to the well-known small-gain theorem, the stability of the closed-loop system can always be guaranteed.
Numerical example

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Consider the simple plant $\dot{x}(t) = x(t) + u(t - 1)$ with the control law

$$u(t) = -(1 + \lambda_d) \left( e^1 \cdot x(t) + \int_0^1 e^\zeta u(t - \zeta) d\zeta \right) + r(t),$$

where $r(t)$ is the reference. The distributed delay is

$$v(t) = \int_0^1 e^\zeta u(t - \zeta) d\zeta,$$

and the $s$-domain equivalent is

$$Z(s) = \frac{1 - e^{1-s}}{s - 1}. $$
The implementation $Z_r$

$$Z_r(s) = \sum_{k=0}^{N-1} \left( \frac{2-\epsilon s}{2-2\epsilon + \epsilon s} \right)^k \frac{2\epsilon}{2-2\epsilon + \epsilon s}$$

with $\epsilon = 1 - e^{-\frac{1}{N}}$. Since $A = 1$ has no non-real eigenvalues, $Z_r$ is always stable, even for $N = 1$. 
Comparative studies

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure}
\caption{Comparison of implementation error for different methods.}
\end{figure}

\textbf{N} = 5
System response

\[ r(t) = 1(t) \]

\[ \lambda_d = 1 \]

\[ u = -(1+\lambda_d) \left( e^1 \cdot x + v \right) + r \]

\[ v = Z_r \cdot u \]

- \( N = 1 \): unstable
- \( N = 2 \): stable but slightly oscillatory
- \( N = 5 \): very close to the ideal response
- All the stable responses guarantee the steady-state performance.
Summary

A rational implementation has been proposed for distributed delay based on the bilinear transformation.

The implementation consists of a series of bi-proper nodes cascaded with a low-pass node.

The $H^\infty$-norm of the implementation error approaches 0 when the number $N$ of nodes goes to $\infty$ and the stability of the closed-loop system can always be guaranteed.

The steady-state performance of the system is guaranteed.

It does not involve any extra parameter to choose apart from the number $N$ of the nodes. In particular, no parameter for a low-pass filter is needed to choose.

Simulation examples and comparisons are given.
Acknowledgment

This work was supported by EPSRC under grant No. EP/C005953/1.