Multicast Capacity for Multi-Hop Multi-Channel Multi-Radio Wireless Networks

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ABSTRACT

Assume that n wireless nodes are randomly deployed in a square region with side-length a and all nodes have the uniform transmission range r and uniform interference range $R = \Theta(r)$. Each node is equipped with ϕ interfaces. There are $C = O(\min(nr^2/a^2, \log n))$ channels of equal bandwidth $\frac{W}{C}$ available. We consider a random (C, g) channel assignment where each node may switch between a preassigned random subset of g channels (with $g \geq \phi$). In this paper, we study the multicast capacity of such a random wireless network, where for each node v_i, we *randomly* pick $k-1$ nodes from the other $n-1$ nodes as the receivers of the multicast session rooted at node v_i. We derive matching asymptotic upper bounds and lower bounds on multicast capacity. We show that the **per-flow** multicast capacity is $\Theta(W\sqrt{\frac{P_{rnd}}{n \log n}} \cdot \frac{1}{\sqrt{k}})$ when $k = O(\frac{P_{rnd} \cdot n}{\log n})$, where P_{rnd} denotes the probability that two nodes share at least one channel. Our bounds unify the previous capacity bounds on unicast (when $k = 2$) by Bhandari and Vaidya [3] for multi-channel multi-radio networks.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design**]: Wireless communication, Network topology; G.2.2 [**Graph Theory**]: Network problems, Graph algorithms

General Terms

Algorithms, Design, Theory

Keywords

Wireless ad hoc networks, capacity, multicast, multichannel.

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1. INTRODUCTION

We often need estimate the achievable throughput when we randomly deploy n wireless nodes in a given region. The main purpose of this paper is to study the *asymptotic capacity* of large scale random wireless networks when we choose the *best* protocols for all layers. As in the literature, we will mainly consider one type of networks, large scale *random networks*, where a large number of nodes

are randomly placed in the deployment region. We will study the capacity of a given wireless network where the nodes positions are given a priori, and how the capacity of wireless networks scale with the number of nodes in the networks (when given a fixed deployment region), scale with the transmission range, scale the size of the deployment region (when given a fixed deployment density) for multicast, or scale with the number of receivers.

We assume that a set of n wireless nodes $V = \{v_1, v_2, \cdots, v_n\}$ are *randomly* distributed (with uniform distribution) in a square region with a side-length a and all nodes have the same transmission range r. For most results presented in this paper, we assume that values of a and r are selected such that the resulted network will be connected with high probability (*w.h.p.*). The results derived under this model also imply the same results for the dense model, when n nodes are distributed in a fixed region (such as a unit square by a proper scaling) and the uniform transmission range of all nodes are selected as the critical transmission range (CTR) to get a connected network with high probability. We focus on multi-hop multi-channel multi-radio (MCMR) large scale wireless ad hoc networks. There are $C = O(\min(nr^2/a^2, \log n))$ channels of equal bandwidth $\frac{W}{C}$ available. Each node is equipped with ϕ interfaces. We further consider a random (C, g) channel-assignment where each node may switch between a pre-assigned random subset of g channels (with $g \geq \phi$).

We study the *multicast capacity* of a random wireless network, which generalizes both the unicast capacity [5], broadcast capacity [7, 15], and unicast capacity for MCMR [3] for random networks. Assume that a subset $\mathcal{S} \subseteq V$ of $n_s = |\mathcal{S}|$ nodes will serve as the source nodes of n_s multicast sessions. Most results in this paper assume that $\mathcal{S} = V$. Each node $v_i \in \mathcal{S}$ has a set of randomly chosen $n_d = k-1$ destination nodes to which it wishes to send data at an arbitrary data rate λ_i. The multicast capacity of a random network is defined as $\Lambda_k(n) = \sum_{i=1}^{n} \lambda_i$ when there is a schedule of transmissions and selection of channels such that all multicast flows will be received by their destination nodes successfully within a finite delay.

Our Main Contributions: We propose two regimes for multicast capacity in terms of k. We derive matching analytical upper bounds and lower bounds on multicast capacity of a random wireless network. Assume that the side-length a of the deployment square and the transmission range r are selected such that the network is connected almost surely. We show that the aggregated multicast capacity of n random multicast flows is

$$\Lambda_k(n) = \begin{cases} \Theta(\sqrt{\frac{P_{rnd} \cdot n}{\log n}} \cdot \frac{W}{\sqrt{k}}) & \text{when } k = O(\frac{n}{\log n}), \\ \Theta(W) & \text{when } k = \Omega(\frac{n}{\log n}) \end{cases} \quad (1)$$

Here $P_{rnd} = 1 - \prod_{i=0}^{g-1}(1 - \frac{g}{C-i})$ is the probability that two nodes will have a common channel to communicate. Notice that here $P_{rnd} = 1$ when $g > \frac{C}{2}$. Our bounds unify the previous capacity bounds on unicast (when $k = 2$) by Gupta and Kumar [5] the capacity bounds on broadcast (when $k = n$) in [7, 15] when single-channel is used; and the unicast capacity by Bhandari and Vaidya [3] when multi-channel and single-radio is used.

Consequently, the per-flow multicast capacity $\lambda_k(n)$ of n multicast sessions (with $k-1$ receivers per multicast session) is

$$\lambda_k(n) = \begin{cases} \Theta(\sqrt{\frac{P_{rnd}}{n \log n}} \cdot \frac{W}{\sqrt{k}}) & \text{when } k = O(\frac{n}{\log n}), \\ \Theta(\frac{W}{n}) & \text{when } k = \Omega(\frac{n}{\log n}) \end{cases} \quad (2)$$

The above capacity bounds are implied by a more general result for the following network setting there are n_s multicast sessions, each with $k-1$ randomly selected receivers from V, and the transmission range r and side-length a of the deployment square satisfying that the resulted random network is connected with high probability. Generally we prove that the aggregated multicast capacity of n_s multicast sessions is

$$\Lambda_k(n) = \begin{cases} \Theta(\frac{a}{r} \cdot \frac{W}{\sqrt{k}}) & \text{when } k = O(\frac{a^2}{r^2}) \\ \Theta(W) & \text{when } k = \Omega(\frac{a^2}{r^2}) \end{cases} \quad (3)$$

and the per-source multicast capacity of n_s multicast sessions is

$$\lambda_k(n) = \begin{cases} \min(W, \Theta(\frac{a}{r} \cdot \frac{W}{n_s\sqrt{k}})) & \text{when } k = O(\frac{a^2}{r^2}) \\ \Theta(\frac{W}{n_s}) & \text{when } k = \Omega(\frac{a^2}{r^2}) \end{cases} \quad (4)$$

2. NETWORK MODEL

We assume that there is a set $V = \{v_1, v_2, \cdots, v_n\}$ of n communication terminals deployed in a region Ω. We mainly focus on the scenario in which Ω is a square with side length a. Every wireless node has a uniform transmission range r such that a node u can successfully receive the signal sent by node v if and only if $\|u - v\| \leq r$. The complete communication graph is a undirected graph $G = (V, E)$, where E is the set of communication links. Each node is equipped with ϕ interfaces.

To schedule two links at the same time slot, we must ensure that the schedule will avoid interference. Several different interference models have been used to model the interferences in wireless networks. In this paper, we mainly focus on the protocol interference model. We assume that each node v_i has a constant interference range R. Any node v_j will be interfered by the signal from v_k if $\|v_k - v_j\| \leq R$ and node v_k is sending signal to some node other than v_j. We always assume that the interference range R is within a small constant factor of the transmission range r, *i.e.*, $R = \Theta(r)$.

There are C available channels of bandwidth $\frac{W}{C}$ each. We focus on the case where the total number of available channels $C = O(\min(nr^2/a^2, \log n))$. We believe this is justified because in large scale deployments, the number of nodes will typically be much larger than the number of available channels. Besides, when $C = \Omega(\min(nr^2/a^2, \log n))$, there is a huge capacity degradation even with unconstrained channel switching, thus making channelization an increasing liability, and constrained switching can only lead to additional degradation, and potentially unacceptable performance. In this paper, we use random (C, g) channel assignment, in which, a node is assigned a subset of g channels ($\phi \leq g \leq C$) uniformly at random from the set of all possible channel subsets $\{f_1, f_2, \cdots, f_C\}$ of size C.

We assume that each node v_i could serve as the source node for some multicast. For each node v_i, we randomly select $k-1$ points $P_i = \{p_{i,1}, p_{i,2}, \cdots, p_{i,k}\}$ from the deployment region. For each point, we select the nearest node from V as the receiver. We then get $k-1$ nodes, say $U_i \subseteq V - \{v_i\}$, from the remaining $n-1$ nodes as the receivers of multicast session using v_i as the source node. Assume that node v_i will send data to these receivers U_i with a data rate λ_i. Notice that when the receivers are far away from the source node, we need multiple intermediate nodes to relay the data for v_i. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n)$ be the *rate vector* of the multicast data rate of all multicast sessions. Given a set of n_s multicast sessions with the set of source nodes $\mathcal{S} \in V$, let $\lambda_{\mathcal{S}} = \{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_{n_s}}\}$ be the vector of data rates of all sources in $\mathcal{S}$. Then $\lambda_k(n) = \min_i \lambda_i$ is called the minimum per-flow multicast capacity, $\Lambda_k(n) = \sum_i \lambda_i$ is called the total multicast capacity, and $\sum_i \lambda_i / n_s$ is called the average per-flow multicast capacity.

Useful Known Results: Throughput this paper, we will repeatedly use the following results from probability theory literature.

LEMMA 1 (CHEBYSHEV'S INEQUALITY). *For a variable X, **Pr** $(|X - \mu| \geq A) \leq \frac{\text{Var}(X)}{A^2}$, where $\mu = E(X)$, $\text{Var}(X)$ is the variance of X, and $A > 0$.*

LEMMA 2 (LAW OF LARGE NUMBERS). *Consider n uncorrelated variables X_i, $1 \leq i \leq n$ with same expected value $\mu = E(X_i)$ and variance $\sigma^2 = \text{Var}(X_i)$. Let $\overline{X} = \frac{\sum_{i=1}^n X_i}{n}$. $\forall \epsilon > 0$,*

$$\textbf{\textit{Pr}}\left(|\overline{X} - \mu| < \epsilon\right) \geq 1 - \frac{\sigma^2}{n \cdot \epsilon^2}.$$

LEMMA 3 (BINOMIAL DISTRIBUTION). *Consider n independent variables $X_i \in \{0, 1\}$, $p = $ **Pr** $(X_i = 1)$, and $X = \sum_{i=1}^n X_i$.*

$$\textbf{\textit{Pr}}(X \leq \xi) \leq e^{\frac{-2(n \cdot p - \xi)^2}{n}}, \quad \text{when } 0 < \xi \leq n \cdot p.$$

$$\textbf{\textit{Pr}}(X > \xi) < \frac{\xi(1-p)}{(\xi - n \cdot p)^2}, \quad \text{when } \xi > n \cdot p.$$

3. UPPER BOUND

3.1 The upper-bound on a/r

We assume that n wireless nodes V with transmission range r are randomly and uniformly distributed in a square region with side length a. We then review the sufficient condition and the necessary condition proved in [3] on the relations of a and r for connectivity of a random wireless network.

LEMMA 4 (BHANDARI AND VAIDYA [3]). ***Sufficient Condition**: With a random (C, g) channel assignment (when $C = O(\log n)$), if $\pi r^2 \geq \frac{800\pi \log n}{P_{rnd} \cdot n} \cdot a^2$, where $P_{rnd} = 1 - \prod_{i=0}^{g-1}(1 - \frac{g}{C-i})$ denotes the probability that two nodes share at least one channel, then almost surely, the network $G = (V, r)$ is connected where V is a set of n nodes randomly and uniformly distributed in a square of side length a.*

***Necessary Condition**: With a random (C, g) channel assignment (when $C = O(\log n)$), if $\pi r^2 = \frac{\log n + b(n)}{P_{rnd} \cdot n} \cdot a^2$, and $\lim_{n \to \infty} b(n) < +\infty$, then the network $G = (V, r)$ will be disconnected with nonzero probability.*

3.2 Upper Bound Capacity When $k = O(a^2/r^2)$

In the setting of multi-channel multi-radio (MCMR) multi-hop wireless networks, some intermediate nodes in the multicast tree will have to receive the data using a channel and then send the data to its downstream receivers using another channel (or channels). A node that has some randomly selected g channels, it may need to switch to several channels to let all downstream neighboring receivers in a multicast tree get the data.

To study the capacity bound on a random network with C channels, at any time slot t, we decompose the multicast network into C layers: each layer is denoted by $G_i(t) = (V_i(t), E_i(t))$, where $V_i(t)$ is the set of nodes choosing channel f_i for transmission at time-slot t. Notice that since each node has ϕ radios, a node v will appear in at most ϕ layers at any time-instance. At different time-slot instance, a node may choose different channels, thus appearing at different layers. However, based on our channel assignment protocol, the set of layers that a node can appear is pre-determined by the set of g channels it was assigned (its radios will always choose ϕ channels from these g pre-determined channels). We will present an upper bound of total multicast capacity. A trivial upper bound is $\frac{W}{C} \cdot n \cdot \phi$ since there are at most n source nodes on all layers and every source node can only send $\frac{W}{C} \cdot \phi$ bits/sec at most. A better upper bound is $\frac{W}{C} \cdot n \cdot \phi \cdot \frac{1}{k}$ since n nodes on all layers can only receive $\frac{W}{C} \cdot \phi$ bits/sec at most and among all received data, the data from any multicast will have at least k copies at these n nodes.

Clearly, when we multicast from one source node v_i to all its $k-1$ receivers U_i, it is more likely that other nodes will also get a copy of the data. Here, for the purpose of analysis, when a node v sends data to one of its neighboring nodes, *all* nodes (within its interference range and use the same channel) will be *charged* a *copy* of the data. Notice that here a neighboring node w may not be the intended receiver. However, since when v is transmitting, any of such "neighboring" node w *cannot* receive data simultaneously from any other transmitting node due to interference, we will say that node w also gets a copy of the data. Notice that the transmission of a node using channel f_i at time t can *only* affect the transmissions of nodes in the same layer $G_i(t)$. Generally, let $M_{i,f}(t)$ be the number of nodes at layer f and time t that will get a copy of the data when the $k-1$ receivers are randomly selected for each possible source node v_i. Parameter t will be omitted if it is clear from the context. Assume that the multicast session rooted at v_i has data rate λ_i. Then, the number of data copies constraint implies that

$$\sum_{t=1}^{\mathcal{T}} \sum_{i=1}^{n} \lambda_i \sum_{f=1}^{C} M_{i,f}(t) \le \sum_{t=1}^{\mathcal{T}} n \frac{W}{C} \cdot \phi. \quad (5)$$

The right hand side is the total number of bits that can be "received" by *all* nodes using all channels in time period $[1, \mathcal{T}]$ when each node has ϕ radios. When $\lambda = \min_i \lambda_i$, we have

$$\lambda \le \frac{n \cdot \phi \cdot W}{C \cdot \sum_{i,f} M_{i,f}} \quad (6)$$

Thus, the per-flow multicast capacity is upper bounded by $\frac{n \cdot \phi \cdot W}{C \cdot \sum_{i,f} M_{i,f}}$. Thus, to get an upperbound on per-flow multicast capacity, it suffices to get a lower-bound on $\sum_{i,f} M_{i,f}$. Based on this formula, we know that the total multicast capacity (when all multicast sessions have the same data rate) is

$$\Lambda_n(k) \le \frac{n^2 \cdot \phi \cdot W}{C \cdot \sum_{i,f} M_{i,f}}. \quad (7)$$

Let M be the asymptotic lower bound of $\sum_{f=1}^{C} M_{i,f}$, *i.e.*, $M_i = \sum_{f=1}^{C} M_{i,f} \ge M$ almost surely for every i. Obviously, the total multicast capacity $\sum_{i=1}^{n} \lambda_i$ is upper bounded by $\Lambda_n(k) \le \frac{n \frac{W}{C} \cdot \phi}{M} = \frac{nW\phi}{C \cdot M}$. The rest of the subsection is devoted to give a better lower bound on M for analyzing the total multicast capacity, or give a better lower bound on $\sum_{i=1}^{n} \sum_{f=1}^{C} M_{i,f}$ for analyzing the upper-bound on per-flow multicast capacity. To analyze these values, we first study the asymptotic lower bound of the Euclidean length $\|T_i\|$ of a multicast tree T_i.

Bound the data copies: A straightforward lower-bound on the number of nodes (including leaf nodes) needed in a multicast tree spanning k nodes randomly selected in a square of side-length a is $\tau \cdot \sqrt{k} \cdot \frac{a}{r}$ with high probability, because the expected Euclidean length of a multicast tree is at least $\tau \cdot \sqrt{k} \cdot a$. Although this bound on M is much better than bound $M \ge k$ when $k = O(a^2/r^2)$, the bound can be further improved based on the following observation. When nodes on the multicast tree relay data from the source node to receivers, not only its downstream nodes of the multicast tree will receive the data, but also all its neighboring nodes (in communication graph G) will get a *copy* of the data. We will then analyze the number of nodes that will get the copy of the data. Given a multicast tree T, let $\text{Area}(T)$ be the region covered by all transmitting disks of all transmitting nodes (internal nodes) in the multicast tree T. Observe that the leaf nodes do *not* contribute to $\text{Area}(T)$ at all here. Given a forest T, $|\text{Area}(T)| = \bigcup_{i=1}^{C} |\text{Area}(T_i)|$, where T_i denotes the subtrees of T in G_i (all nodes transmitting using channel f_i). Let $X = \|EMST(U)\|$ be the Euclidean length of the EMST tree of a set of k randomly selected nodes U in a square of side-length a.

LEMMA 5 ([12]). *The total edge length, denoted by $\|T\|$, of any multicast tree T spanning k nodes that are randomly placed in a square of side-length a, with high probability, is at least, when $k \to \infty$, $\tau \cdot \sqrt{k} \cdot a$. Here τ is a constant depending on the dimension.*

It was shown in [14] that $\text{Var}(X) \ll a^2 \cdot \log k$. The following lemma was also proved in [12].

LEMMA 6. *[12] For any k nodes U placed in a square region with side-length a, the length of EMST spanning U is at most $2\sqrt{2}\sqrt{k}a$.*

Clearly, the area of $\text{Area}(T)$, denoted by $|\text{Area}(T)|$, is at most $|\text{Area}(T)| \le 2r \cdot \|T\| + k \cdot \pi r^2/2$ where $\|T\|$ is the total Euclidean length of all links in T. The following lemmas were proved in [12].

LEMMA 7. *[12] The area of the region $\text{Area}(T)$, $|\text{Area}(T)|$, is at most $\tau\sqrt{k}a \cdot 2r + k \cdot \pi r^2/2$, and is at least $\frac{\tau\sqrt{k}a \cdot r}{c_0}$ with high probability, when the number of receivers/source nodes $k < \theta_1 \frac{a^2}{r^2}$ for some constant θ_1, and c_0.*

LEMMA 8. *[12] When $k \ge \theta_1 \cdot \frac{a^2}{r^2}$, the area covered by the transmitting disks of a multicast tree is almost surely at least $\varrho_2 \cdot a^2$, thus, $\Theta(a^2)$.*

In other words,

$$|\text{Area}(T)| \ge \begin{cases} \frac{\tau}{c_0}\sqrt{k}ra, & \text{when } k \le \theta_1 a^2/r^2 \\ \varrho_2 a^2, & \text{when } k \ge \theta_1 a^2/r^2 \end{cases} \quad (8)$$

Notice that when $k = \Omega(a^2/r^2)$, $\sqrt{k}ra = \Omega(a^2)$; when $k = O(a^2/r^2)$, $\sqrt{k}ra = O(a^2)$. Consequently, we have a general bound for $|\text{Area}(T)|$: almost surely the area covered by all transmitting disks of a multicast tree for k receivers/source

$$|\text{Area}(T)| = \Omega(\min(a^2, \sqrt{k}ra)). \quad (9)$$

We then analyze the number of nodes that will get a copy of the multicast data from one multicast session in the setting of multi-channel multi-radio (MCMR) networks. Notice that unlike the single-channel single-radio networks studied in [12,13], in MCMR networks, a transmission by a node v will *only* exclude nodes within distance r from transmitting using the *same* channel. Intuitively, when every node randomly selects g channels from C channels, the *expected* number of nodes that are within its transmission range and share the same channel with a transmitting node v (using a specific channel) is $\frac{\phi n \cdot r^2}{a^2 \cdot C}$. Since $C = O(nr^2/a^2)$, $\frac{\phi n \cdot r^2}{a^2 \cdot C} = \Omega(\phi)$.

LEMMA 9. *Given a multicast tree T rooted at node v_i, with high probability, the number $M_i = \sum_{f=1}^{C} M_{i,f}$ of nodes that will get a copy of the multicast data is at least*

$$M_i \geq \begin{cases} \frac{\tau\phi}{2c_0 C} \cdot \frac{n\sqrt{k}r}{a} & \textit{when } k \leq \theta_1 \frac{a^2}{r^2} \\ \frac{\varrho_2 \phi n}{2C} & \textit{when } k \geq \theta_1 \frac{a^2}{r^2} \end{cases} \quad (10)$$

For a node v_j, let variable $X_{i,j,f} \in \{0,1\}$ be an indicator whether node v_j is using channel f and it is falling inside the transmission range of a transmitting node of the multicast tree T for v_i. Clearly, $X_{i,j,f} = 1$ if and only if

1. node v_j is using channel f, which has probability $\frac{\phi}{g} \cdot \frac{g}{C} = \frac{\phi}{C}$ since every node has ϕ radios and it will randomly select g different channels from total C channels; and
2. node v_j is inside the transmitting disk of a node in T, which has probability $\frac{\text{Area}(T)}{a^2}$.

Let $p = \frac{\phi}{C} \cdot \frac{\text{Area}(T)}{a^2} = \mathbf{Pr}(X_{i,j,f} = 1)$. Then variable $M_{i,f} = \sum_{j=1}^{n} X_{i,j,f}$ is the number of nodes that will get a copy of the multicast data from v_i using channel f, and $M_i = \sum_{j=1}^{n} \sum_{f=1}^{C} X_{i,j,f}$ is the number of nodes that will get a copy of the multicast data from v_i in *all* C layers. Then $E(M_{i,f}) = n \cdot \frac{\phi}{C} \cdot \frac{\text{Area}(T)}{a^2}$, and $E(M_i) = n \cdot C \cdot \frac{\phi}{C} \cdot \frac{\text{Area}(T)}{a^2}$. Using Lemma 3, we can easily show that

$$\mathbf{Pr}\left(M_i = \sum_{f=1}^{C} \sum_{j=1}^{n} X_{i,j,f} \leq \frac{nCp}{2} \right) \leq e^{-nC \cdot p^2/2} \quad (11)$$

Notice that $C = O(\min(nr^2/a^2), \log n)$, and *w.h.p.*, $\text{Area}(T) = \min(\varrho_2 a^2, \frac{\tau}{c_0}\sqrt{k}a \cdot r)$. Thus we have $nC \cdot p^2 \geq \frac{n\phi^2 \varrho_2^2}{C} > 2 \ln n$ when $k > \theta_1 a^2/r^2$ and $nC \cdot p^2 \geq \frac{\phi^2 \tau^2 k}{c_0} \geq 2 \ln n$ when $\frac{2c_0^2 \ln n}{\phi^2 \tau^2} \leq k < \theta_1 a^2/r^2$. Then, with high probability (at least $1 - \frac{1}{n}$), the number of nodes M_i that will "receive" a copy of the multicast data of flow i is at least

$$M_i \geq n \cdot \frac{\phi}{C} \cdot \frac{\text{Area}(T)}{2a^2} \geq \min(\frac{\varrho_2 \phi n}{2C}, \frac{\tau\phi}{2c_0 C} \cdot \frac{n\sqrt{k}r}{a}) \quad (12)$$

This finishes the proof.

Notice that the preceding lemma requires that $nCp^2 \geq 2 \ln n$ to guarantee that the statement holds with a high probability. Then we have

THEOREM 10. *The multicast capacity with $k-1$ receivers for n nodes that are randomly and uniformly deployed in a square with side-length a is at most*

$$\Lambda_k(n) \leq \begin{cases} \frac{2c_0}{\tau} \frac{a}{r} \cdot \frac{W}{\sqrt{k}}, & \textit{when } k = O(\frac{a^2}{r^2}) \\ \frac{2W}{\varrho_2}, & \textit{when } k = \Omega(\frac{a^2}{r^2}) \end{cases} \quad (13)$$

Based on lemma 4, we know that a necessary condition for connectivity is $r/a = \Omega(\sqrt{\frac{\log n}{P_{rnd} n}})$, then we have

THEOREM 11. *The aggregated multicast capacity for a random network of n nodes is at most*

$$\Lambda_k(n) \leq \begin{cases} O(\sqrt{\frac{P_{rnd} n}{\log n}} \cdot \frac{W}{\sqrt{k}}), & \textit{when } k = O(\frac{a^2}{r^2}) \\ O(W), & \textit{when } k = \Omega(\frac{a^2}{r^2}) \end{cases} \quad (14)$$

Similarly, we have

THEOREM 12. *The maximum per-flow multicast capacity of n_s random multicast flows for a random network of n nodes is at most*

$$\lambda_k(n) \leq \begin{cases} O(\sqrt{\frac{P_{rnd} n}{\log n}} \cdot \frac{W}{n_s\sqrt{k}}), & \textit{when } k = O(\frac{a^2}{r^2}) \\ O(\frac{W}{n_s}), & \textit{when } k = \Omega(\frac{a^2}{r^2}) \end{cases} \quad (15)$$

when $n_s \geq 2 \ln n$.

4. LOWER BOUND

In the previous section, we have derived upper bounds on the multicast capacity $\Lambda_k(n)$ and $\lambda_k(n)$. In this section, we will derive asymptotically matching lower bounds on the multicast capacity $\Lambda_k(n)$ and $\lambda_k(n)$. Specifically, we will provide a multicast scheme and prove that the multicast capacity achieved by our scheme matches the asymptotic upper bounds.

4.1 Good Approximation of MCDS

Our multicast scheme is based on a good minimum connected dominating set (MCDS). We construct an approximation of MCDS as follows. First, partition the region into squarelets, each of side-length $r/\sqrt{5}$. Thus, any two nodes from 2 adjacent squarelets (sharing a common side) will be able to communicate with each other directly. Randomly select one node from each squarelet. Clear the set of selected nodes is a dominating set. Then, we connect every pair of dominating nodes that are at most 3 hops away using the least-hop path. The nodes on such least-hop paths will be marked as *connectors*. The set of all dominating nodes and connectors will form a connected dominating set. It can be shown that we can schedule the transmissions of all nodes in CDS in a constant time-slots without interference, and the CDS is a length spanner.

The degree of any node on CDS is bounded by a constant $D_1 \leq (3\sqrt{5})^2 \cdot 3$. Using this property, it is easy to show that the number of nodes in the CDS constructed above that can interfere with any node in the CDS is at most a constant Δ. Using the area argument, we can show that

$$\Delta \leq \frac{\pi \cdot (R + 2r)^2 \cdot (D_1 + 1)}{\pi \cdot r^2} = (1 + \frac{2R}{r})^2 (D_1 + 1).$$

This property ensures that we can schedule the transmissions of all nodes in CDS by a TDMA manner such that all nodes will be able to transmit at least once in every $\Delta + 1$ time slots. Notice that here Δ is a constant.

For any two nodes u and v in the network, if $\|u - v\| > r$, then the shortest path connecting u and v via the CDS (constructed above) has length at most 5 times the length of the shortest path connecting them in the original random communication graph $G = (V, E)$ (proof is similar to Lemma 5 of [2]). Notice that here when u (or v or both) is not in the CDS, we will first connect u (or v or both) to one of its dominators (say u' and v') in the CDS. Then we find the shortest path connecting these corresponding dominators u' and v' in the CDS.

4.2 Partition Square Using Squarelets

Our multicast scheme is based on a good approximation of a minimum connected dominating set (MCDS) of a random network.

First, partition the region into squarelets, each of side-length $r/\sqrt{5}$. Thus, any two nodes from 2 adjacent squarelets (sharing a common side) will be able to communicate with each other directly. Randomly select one node from each squarelet. Clearly the set of selected nodes is a dominating set. If every squarelet has a node inside it, obviously, the set of selected nodes will form a connected dominating set (CDS).

Notice that, it is possible that, for some squarelet, there is no node with at least one common channel inside, and thus, we cannot find a multicast tree $MT(U_1')$ later by Algorithm 1. We show that this almost surely cannot happen.

LEMMA 13. *There is a sequence of* $\delta(n) \to 0$ *such that*

$$\textbf{Pr}\,(\textit{Given channel } i\textit{, every squarelet contains some nodes using } i\,)$$

$$\geq 1 - \delta(n)$$

Let $\mathcal{C}$ be the class of axis-aligned squares of side-length $\frac{r}{\sqrt{5}}$. Notice that the probability that a node fall in such a square is $\frac{r^2}{5} \cdot \frac{1}{a^2} = \frac{r^2}{5a^2}$. Recall that, to have a connected network, we almost surely have $r/a \geq \sqrt{\frac{\log n}{nP_{rnd}}} \geq \sqrt{\frac{\log n}{n\pi}}$. It is easy to show that the VC-dimension of $\mathcal{C}$ is at most 4 (it is at least 3). Hence, for all squarelets S, $\textbf{Pr}\left(\sup_{S\in\mathcal{C}}\left|\frac{\#\text{ of nodes in } S}{n} - \frac{r^2}{5a^2}\right| \leq \epsilon(n)\right) > 1-\delta(n)$ whenever

$$n \geq \max\left\{\frac{32}{\epsilon(n)}\cdot\log\frac{13}{\epsilon(n)}, \frac{4}{\epsilon(n)}\log\frac{2}{\delta(n)}\right\}. \quad (16)$$

This condition 16 is satisfied when $\epsilon(n) = \frac{32\log n}{n}, \delta(n) = \frac{2}{n}$. Thus,

$$\textbf{Pr}\left(\sup_{S\in\mathcal{C}}\{\#\text{ of nodes in } S \geq \frac{nr^2}{5a^2} - n\cdot\epsilon(n)\}\right) > 1-\delta(n)$$

Thus, if we choose r and a such that

$$r \geq 14\cdot a\cdot\sqrt{\frac{\log n}{n}} \quad (17)$$

then $\frac{nr^2}{5a^2} - n\cdot\epsilon(n) \geq 7\log n$. Consequently, we have

$$\textbf{Pr}\,(\forall\text{ squarelet } S, \#\text{ of nodes in } S \geq 7\log n) > 1 - \frac{2}{n}$$

Thus,

$$\textbf{Pr}\,(\forall\text{ squarelet } S,\text{ given any channel } i \text{ there are some nodes in } S \text{ using } i)$$

$$\geq 1 - (1 - \frac{g}{C})^{7\log n}$$

The theorem then follows.

For each node on the CDS, we show that every node can be scheduled to transmit once every Δ time-slots, where constant Δ depending only on R and r. For each node v, consider a node u whose transmission will interfere with the transmission of node v. Clearly node u will be completely inside the disk centered at v with radius $R+r$. Thus, the squarelet containing u must be inside the disk centered at v with radius $R + r + \frac{\sqrt{2}}{\sqrt{5}}r < R + 2r$. Let Δ be the maximum number of nodes in CDS whose transmission will interfere with the transmission of a node v in CDS. Using the area argument, we can show that $\Delta \leq \frac{\pi\cdot(R+2r)^2}{r^2/5} = 5\pi(2+\frac{R}{r})^2$. This property ensures that we can schedule the transmissions of all nodes in CDS by a TDMA manner such that all nodes will be able to transmit at least once in every Δ time slots. Notice that here Δ is a constant.

4.3 Lower Bound Capacity When $k = O(a^2/r^2)$

When the number of receivers, plus the source node, k is only $O(\frac{a^2}{r^2})$, we will construct a multicast tree from CDS. Consider an instance of a random network $G = (V, E)$ and also an instance of multicast with v_i as the source node. Let $P_i = \{p_{i,1}, p_{i,2}, \cdots, p_{i,k}\}$ be the set of randomly and independently selected points used to find the terminals $U_i = \{v_{i,1}, v_{i,2}, \cdots, v_{i,k}\}$. Recall that $v_{i,j}$ is the closest node to point $p_{i,j}$. Let $U_i' = U_i \cap \{v_i\}$. We then construct a multicast structure as Algorithm 1.

Algorithm 1 Multicast routing for nodes U_i

1: We partition the deployment square into squarelets, each with side length $r/\sqrt{5}$. Each squarelet is denoted by (i, j) when it is the ith column and jth row.
2: We build an Euclidean spanning tree, denoted as $EST(P_i)$, connecting points in P_i.
3: For each link uv in the tree $EST(P_i)$, assume that u and v are inside squarelet (i_u, j_u) and squarelet (i_v, j_v) respectively. Find a point w in squarelet (i_v, j_u) (or squarelet (i_u, j_v)), *i.e.*, uwv is a Manhattan path connecting u and v. The resulted structure by uniting all such paths for all links in $EST(P_i)$ will serve the routing guideline for multicast.
4: For each edge uw in $EST(P_i)$, find a node in each of the squarelets that are crossed by line uw. We connect these nodes in sequence to form a path, denoted as $\mathbf{P}(u, v)$, connecting points u and v. If the resultant structure is not a tree, we could remove the cycles that do not contain nodes from P_i. Denote the resulted tree as $MT(P_i)$.
5: For each receiver $v_{i,j}$, if it is not inside the squarelet **s** containing point $p_{i,j}$, let $v_{i,j}'$ be the node selected inside the squarelet **s**. Notice that, such $v_{i,j}'$ exists for every squarelets, with probability at least $1 - 2/n$. Node $v_{i,j}'$ then relay the data to node $v_{i,j}$ (the relay takes at most 2 hops). The final tree (including these additional relays) are called multicast tree $MTR(U_i)$.

To show that the preceding routing method achieves the asymptotically optimum multicast capacity, we first show that the total number of data copies of a multicast bit is at most $O(\frac{r\cdot\sqrt{k}\cdot n}{a}\cdot\frac{\phi}{C})$, which will be derived based on the upper bound on the area covered by all transmission disks in the multicast tree $MT(U_1')$. The following lemma was proved in [12].

THEOREM 14. *[12] With high probability, the total Euclidean length of the multicast tree* $MT(U_1')$ *is within a constant* c_5 *factor of* $\|EMST(U_1')\|$, *i.e.,* $\|MT(U_1')\| \leq c_5\|EMST(U_1')\|$.

Consequently, we have Euclidean length $\|MT(U_1')\| < c_5\eta\sqrt{k}\cdot a$ with high probability since $\|EMST(U_1')\| \leq \eta\sqrt{k}\cdot a$ for $\eta = 2\sqrt{2}$ (see Lemma 6). Here we denote the region covered by all transmission disks of all internal nodes in $MT(U_1')$ as $\text{Area}(T)$ and the number of nodes lying inside $\text{Area}(T)$ as Z. We then show that with high probability, the multicast capacity achieved using above routing approach is within a constant factor of the asymptotic optimum. We essentially show that, with high probability, the number Z of nodes that will receive a *copy* of the multicast data is within $2E(Z)$.

THEOREM 15. *The total multicast capacity* $\Lambda_k(n)$ *achievable by all multicast flows is at least* $c_6\frac{a\cdot W}{r\sqrt{k}}$, *when* $k \leq \theta_1\frac{a^2}{r^2}$ *and* $a/r \leq \sqrt{\frac{P_{rnd}n}{\log n}}$. *Here* c_6 *is a constant.*

Consider a set of receivers U_1 for source node v_1. Let tree T be the multicast tree $MT(U_1')$ constructed above. Let $X_i \in \{0, 1\}$

be an indicator variable whether the ith node v_i will fall inside the region $\text{Area}(T)$ for a multicast tree T. Clearly $p = \mathbf{Pr}(X_i = 1) = \frac{|\text{Area}(T)|}{a^2}$. Notice that the area of $\text{Area}(T)$ is at most $2r \cdot \|T\| + k\pi r^2/2$, and edge length $\|T\| \le c_5\eta \cdot \sqrt{k} \cdot a$ with high probability. Obviously, $X = \sum_{i=1}^{n} X_i$ is the number of nodes falling inside the region $\text{Area}(T)$, and X is binomial distribution. Using Lemma 3, we have

$$\begin{aligned}
&\mathbf{Pr}\left(X > |\text{Area}(T)| \cdot \frac{2n}{a^2}\right) \\
\le\ & \frac{|\text{Area}(T)| \cdot \frac{2n}{a^2} \cdot (1 - \frac{|\text{Area}(T)|}{a^2})}{(|\text{Area}(T)| \cdot \frac{2n}{a^2} - \frac{n \cdot |\text{Area}(T)|}{a^2})^2} \\
=\ & \frac{2[1 - \frac{|\text{Area}(T)|}{a^2}]}{|\text{Area}(T)| \cdot \frac{n}{a^2}} \le \frac{2a^2}{n \cdot |\text{Area}(T)|} \le \frac{2c_0 \cdot a^2}{n \cdot \tau\sqrt{k} \cdot a \cdot r} \\
=\ & \frac{a}{r}\frac{1}{n\sqrt{k}}\frac{2c_0}{\tau} \le \frac{1}{\sqrt{n \cdot k \cdot \ln n}} \cdot \frac{2c_0\sqrt{P_{rnd}}}{\tau}
\end{aligned}$$

The last inequality comes from the assumption that $a/r \le \sqrt{\frac{P_{rnd}n}{\log n}}$. The second to last inequality comes from Lemma 7 that $|\text{Area}(T)| \ge \frac{\tau\sqrt{k} \cdot a \cdot r}{c_0}$. Consequently,

$$\mathbf{Pr}\left(X \le |\text{Area}(T)| \cdot \frac{2n}{a^2}\right) \ge 1 - \frac{1}{\sqrt{n \cdot k \cdot \ln n}}\frac{2c_0\sqrt{P_{rnd}}}{\tau}$$

. Thus, using Lemma 3, similar to the proof of Lemma 9 (by defining variable $X_{i,j,f}$), we can show that the number of nodes that will get a copy of the data for multicast within nodes U_1', with high probability, is at most

$$\begin{aligned}
|\text{Area}(T)| \cdot \frac{2n\phi}{a^2 \cdot C} &\le (2r \cdot \eta c_5\sqrt{k}a + k \cdot \pi \cdot r^2/2) \cdot \frac{2n\phi}{a^2 \cdot C} \\
&\le (4c_5\eta + \pi\sqrt{\theta_1}) \cdot n\sqrt{k} \cdot \frac{r}{a} \cdot \frac{\phi}{C}.
\end{aligned}$$

The last inequality comes from $k \le \theta_1 \frac{a^2}{r^2}$.

Recall that, by performing multicast based on CDS structure, we can guarantee that, for all nodes in any layer, each CDS node will be able to transmit once every Δ time-slots. This implies that the total bits/sec received by all nodes in *all* C layers is at least $n\phi \cdot \frac{W/C}{\Delta+1}$. Consequently, the multicast capacity is at least

$$\frac{n\phi \cdot \frac{W/C}{\Delta}}{(4c_5\eta + \pi\sqrt{\theta_1}) \cdot n\sqrt{k} \cdot \frac{r}{a} \cdot \frac{\phi}{C}} = \frac{1}{(4c_5\eta + \pi\sqrt{\theta_1}) \cdot \Delta} \cdot \frac{a \cdot W}{r\sqrt{k}}$$

This finishes the proof by setting $c_6 = \frac{1}{(4c_5\eta + \pi\sqrt{\theta_1}) \cdot \Delta}$.

Observe that the correctness of Theorem 15 relies on the fact that $a/r \le \sqrt{\frac{P_{rnd} \cdot n}{\log n}}$ and $k \le \theta_1 a^2/r^2$. Consequently, by letting $a/r = \sqrt{\frac{P_{rnd} \cdot n}{\log n}}$, based on Theorem 15, we have

COROLLARY 16. *The multicast capacity for a random network of n nodes, when $k < \theta_1 \cdot a^2/r^2$, is at least*

$$\Lambda_k(n) \ge c_6 \cdot \frac{\sqrt{nP_{rnd}}}{\sqrt{\log n} \cdot \sqrt{k}} \cdot W = \Omega(\frac{\sqrt{nP_{rnd}}}{\sqrt{\log n} \cdot \sqrt{k}} \cdot W).$$

Consequently, the multicast capacity per node (with n sources) is

$$\lambda_k(n) = \frac{\Lambda_k(n)}{n} = \Omega(\frac{\sqrt{P_{rnd}}}{\sqrt{n \log n} \cdot \sqrt{k}} \cdot W).$$

Observe that the correctness of Theorem 15 requires the following two additional properties of our routing scheme

1. The traffic "load" on any single channel of *every* routing squarelet is no more than a constant factor of $\frac{W}{C}$ bits/sec (at most $\frac{W/C}{\Delta}$ in this paper), due to the requirement of TDMA node scheduling.
2. At least a constant fraction of nodes V that will send the multicast data or is within the transmission range of some transmitting nodes. This is required for using $n\phi \cdot \frac{W/C}{\Delta}$ as an approximation of the total bits "received" by all nodes per unit time. This condition is clearly satisfied when every node could serve as the multicast source. We will prove that it is still true when there are n_s multicast sessions and n_s satisfies some condition.

We will first prove that, for any squarelet $\mathbf{s}$, with high probability, the *traffic load* (total data rates) assigned to radios in $\mathbf{s}$ is at most $W/\Delta C$. Given a squarelet, we define its *flow-load* as the *total number* of multicast sessions that will be routed through nodes inside this squarelet. We show that under our routing algorithm, for any squarelet, with high probability, its single channel flow-load is no more than $\Theta(\frac{\sqrt{kn\log n}}{C})$. To prove our claim, we first study a simple unicast case. Consider a grid of $L \times L$ squarelets. Consider a specific squarelet $\mathbf{s}$ that is of ith row and jth column in the squarelet-grid. Randomly pick two nodes u and v from the grid and connect them via Manhattan routing. Let $X_{\mathbf{s}}$ be a random variable denoting whether the Manhattan routing will use nodes from the squarelet $\mathbf{s}$. Let $p_{\mathbf{s}}(L)$ denote the probability that the Manhattan routing will use nodes from the squarelet $\mathbf{s}$, *i.e.*, $p_{\mathbf{s}}(L) = \mathbf{Pr}(X_{\mathbf{s}} = 1)$. Then

$$p_{\mathbf{s}}(L) = \frac{i-1}{L^2} \cdot \frac{L-i+1}{L} + \frac{j-1}{L^2} \cdot \frac{L-j+1}{L}. \qquad (18)$$

Here $\frac{i-1}{L^2} \cdot \frac{L-i+1}{L}$ (resp. $\frac{j-1}{L^2} \cdot \frac{L-j+1}{L}$) is the probability that squarelet $\mathbf{s}$ is used when u (resp. v)is on the same row (resp. column) as $\mathbf{s}$. It is easy to show that $\frac{2}{L^2} \le p_{\mathbf{s}}(L) \le \frac{2}{L}$.

Let us now study the number of times that a specific squarelet $\mathbf{s}$ is used by our routing structure for multicast. We use the following method to construct spanning tree in step (2) of Algorithm 1.

1. Originally, k nodes P_i form k components;
2. repeat steps (3) and (4) for $p = 1, 2, \cdots, k-1$,
3. for the gth step, partition the deployment square into at most $k - p$ square-shaped-cells, each with side length $\lceil \frac{a}{\lfloor\sqrt{k-p}\rfloor} \rceil$;
4. find a cell that contains two points of P_i that are from 2 different connected components and then connect them using Manhattan routing; merge these two connected components.

LEMMA 17. *[16] Given a squarelet $\mathbf{s}$, the probability that a random multicast flow will be routed via the squarelet $\mathbf{s}$ is at most $c_6\sqrt{k} \cdot \frac{r}{a}$.*

Similarly, using $\frac{2}{L^2} \le p_{\mathbf{s}}(L)$, we can show that

LEMMA 18. *Given a squarelet $\mathbf{s}$, the probability that a random multicast flow will be routed via the squarelet $\mathbf{s}$ is at least $k \cdot \frac{r^2}{5a^2}$.*

Thus, given any squarelet $\mathbf{s}$, the expected number of flows that will be routed through the squarelet $\mathbf{s}$ is at most $c_6 \cdot n_s\sqrt{k}\frac{r}{a}$, given n_s multicast sessions. Notice that, to achieve larger multicast capacity, we will set $\frac{a}{r} = \sqrt{\frac{cn}{\log n}}$ for some constant $0 < c \le 1/160$ (see proof of Lemma 13). Thus, $\mathbf{Pr}(X_{\mathbf{s}} = 1) \le c_6\sqrt{\frac{k \cdot \log n}{cn}}$. Then we have the following lemma

LEMMA 19. *Given n_s multicast sessions, the expected number of multicast routing flows that use a specific squarelet $\mathbf{s}$ is at most $\frac{c_6}{\sqrt{c}} \cdot n_s \cdot \sqrt{\frac{k \cdot \log n}{n}}$. When $n_s = n$, it is at most $\frac{c_6}{\sqrt{c}} \cdot \sqrt{k \cdot n \cdot \log n}$.*

Recall that, the multicast rooted at v_i will first randomly and independently select $k-1$ points P_i'. To use the VC Theorem, we will construct a multicast tree using the union of node v_i and P_i' as P_i, which is the input of Algorithm 1. The data will then be relayed to every node $v_{i,j}$ if it did not receive the data before. Thus, the *points* used to construct the multicast trees $MT(P_i)$ for different source nodes are independently and randomly chosen in the deployment region. Notice that, given k terminals U, the multicast tree $MT(U)$ constructed by Algorithm 1 can be uniquely defined by its terminals U, thus has dimension $2k$. In other words, every point in $2k$-dimensional cube (with side-length a), corresponds to a multicast tree. Given any 2-dimensional axis-aligned square h (not necessarily the squarelet produced by partitioning the deployment region), let set $T(h)$ be the set of multicast trees (equivalently, the set of points in R^{2k} defining these trees) that will intersect the square h (*i.e.*, one of its edges will have point inside h). Let

$$\mathcal{F} = \{T(h) \mid h \text{ is an axis-aligned square with size } \frac{r}{\sqrt{5}}\}.$$

We will show that the VC-dimension of $\mathcal{F}$ is at most $d = \Theta(\log k)$.

To prove this, we first study the VC-dimension of the following system. Let X be the universal set of 2-dimensional segments. For an axis-aligned square h with a fixed side-length, let $X(h)$ be the set of all segments from X that intersect (or is contained inside) the square h. Let

$$\mathcal{S} = \{X(h) \mid h \text{ is an axis-aligned square with size } \frac{r}{\sqrt{5}}\}.$$

Given *any* set of m line segments $\mathcal{L} = \{L_1, L_2, \cdots, L_m\}$, we show that the cardinality of

$$\Pi_{\mathcal{S}}(\mathcal{L}) = \{\mathcal{L}(h) \mid h \text{ is a 2D square with side-length } \frac{r}{\sqrt{5}}\}$$

is polynomial of m.

LEMMA 20. *[16] The cardinality of $\Pi_{\mathcal{S}}(\mathcal{L})$ is at most $2m^2$, where m is the cardinality of $\mathcal{L}$.*

Notice that, for m segments $\mathcal{L}$, the cardinality of $\Pi_{\mathcal{S}}(\mathcal{L})$ is at most $\frac{m^2}{2} \cdot 4 = 2m^2$. It implies that when a set $\mathcal{L}$ with cardinality m is shatterable by $\mathcal{S}$, $2m^2 \geq 2^m$. Thus, $m < 7$. Consequently, the VC-dimension of $\mathcal{S}$ is at most 6.

LEMMA 21. *[16]The VC-dimension of $\mathcal{F}$ is at most $d = \Theta(\log k)$.*

THEOREM 22. *Assume that there are N random multicast sessions. There is a sequence of $\delta(n) \to 0$ such that*

$$\textbf{\textit{Pr}}\left(\forall \text{ squarelet } \mathbf{s}, \text{number of flows using } \mathbf{s} \leq \frac{3\sqrt{c_6}N}{2}\sqrt{k}\frac{r}{a}\right) \geq 1-\delta(n)$$

The terminals to constructed multicast trees are *i.i.d.* variables. Then the multicast trees are *i.i.d.* variables. Thus, we can use the VC-Theorem. Recall that, given a square h, the probability that a multicast tree will cross h is at most $c_6\sqrt{k} \cdot \frac{r}{a}$ (see Lemma 17). Hence, for all squarelets S,

$$\textbf{Pr}\left(\sup_{S\in\mathcal{F}}\left|\frac{\#\text{ of flows using } S}{N} - P(S)\right| \leq \epsilon(n)\right) > 1-\delta(n)$$

whenever

$$N \geq \max\left\{\frac{8d}{\epsilon(n)} \cdot \log\frac{13}{\epsilon(n)}, \frac{4}{\epsilon(n)}\log\frac{2}{\delta(n)}\right\}. \quad (19)$$

Here d is the VC-dimension of $\mathcal{F}$ and $P(S)$ is the probability of a set S, which is at most $c_6\sqrt{k} \cdot \frac{r}{a}$. Thus,

$$\textbf{Pr}\left(\sup_{S\in\mathcal{F}}\frac{\#\text{ of flows using } S}{N} \leq \frac{c_6\sqrt{k}r}{a} + \epsilon(n)\right) > 1-\delta(n)$$

whenever condition (19) is satisfied. Let

$$\epsilon(n) = \frac{c_6\sqrt{k}r}{2a}, \text{ and } \delta(n) = \frac{2}{n}. \quad (20)$$

Let $\frac{a}{r} = \sqrt{\frac{c\cdot n\cdot P_{rnd}}{\log n}}$ for a constant $0 < c \leq 1/160$. Then it suffices that $N \geq$

$$\max\{\frac{8d\sqrt{c}}{c_6}\sqrt{\frac{n}{k\log n}}\cdot\log(\frac{26\sqrt{c}}{c_6}\sqrt{\frac{n\cdot P_{rnd}}{k\log n}}), \frac{8\sqrt{c}}{c_6}\sqrt{\frac{n\cdot P_{rnd}}{k\log n}}\log n\}.$$

When n is sufficiently large, it is sufficient that $N \geq \frac{4d\sqrt{c}}{c_6}\sqrt{\frac{n\log n\cdot P_{rnd}}{k}}$.

Similar to Theorem 22, we can prove the following theorem using Lemma 18.

THEOREM 23. *Assume that there are N random multicast sessions. There is a sequence of $\delta(n) \to 0$ such that*

$$\begin{aligned}&\textbf{\textit{Pr}}\,(\forall \text{ squarelet } \mathbf{s}, \#\text{ of flows using } \mathbf{s} \geq 1)\\ \geq\ & 1-\delta(n)\end{aligned}$$

when $N \geq \Omega(\max\{\log n, \frac{a^2}{kr^2}\})$.

LEMMA 24. *The number of flows that enter any cell on any single channel is $\Theta(\frac{\sqrt{kn\log n}}{C})$ w.h.p.*

In [3], they present an inductive argument to show how to do backbone construction. At each step of the (inductive) construction, they first have a channel-allocation phase, followed by a node-allocation phase. Here we can use almost the same method to allocate the channel and node location, notice that instead of *Straight Line Routing*, we use *Manhattan Routing* to connect two terminal nodes. Lemma 24 follows based on their proof. Notice that condition $N \geq \frac{4d\sqrt{c}}{c_6}\sqrt{\frac{n\log n\cdot P_{rnd}}{k}}$ can always be satisfied as long as we have $n_s = \Omega(\sqrt{\frac{n\log n}{k}} \cdot \log k)$ multicast sessions. Consequently, if we assign data rate

$$\lambda = \frac{W}{2\Delta(\frac{3\sqrt{c_6}N}{2}\sqrt{k}\frac{r}{a})} = O(\frac{W}{\log n}) \quad (21)$$

to each of the N multicast sessions (with random source node and random terminal points), with probability at least $1-\frac{2}{n}$, the total data rate that will be routed through *every* squarelet is at most $\frac{W}{2\Delta\cdot C}$. Recall that, for a multicast rooted at v_i, our routing will first send data to squarelets containing points P_i. Then for $1 \leq j \leq k-1$, we need forward the data from the squarelet containing the point $p_{i,j}$ to the nearest node $v_{i,j}$ respectively. Notice that we have proved that, for *every* squarelet, with probability at least $1-\frac{2}{n}$, there are at least $\Theta(\log n)$ nodes inside. Thus, with probability at least $1-\frac{2}{n}$, every data can be transferred to some node inside the squarelet. Consequently, counting this last-hop relay, with probability at least $1-\frac{2}{n}$, the total data rate *every* squarelet has to route on single channel is at most $2\frac{W}{2\Delta\cdot C} = \frac{W}{\Delta\cdot C}$. Thus, these flows can be supported by a TDMA scheduling.

Consequently, we have the following theorem

THEOREM 25. *Assume $k \leq \theta_1\frac{a^2}{r^2}$, there are n_s random multicast sessions and $n_s \geq \frac{4d\sqrt{c}}{c_6}\sqrt{\frac{n\log n}{k}}$. With probability at least*

$(1-\frac{2}{n})^2 \geq 1-\frac{4}{n}$, *the achievable aggregated multicast capacity is at least*

$$\Lambda_k(n) = \frac{W}{3\sqrt{c_6}\Delta\sqrt{k}\frac{r}{a}} = \Theta(\frac{W}{\sqrt{k}} \cdot \frac{a}{r}). \quad (22)$$

THEOREM 26. *Assume $k \leq \theta_1 \frac{a^2}{r^2}$, there are n_s random multicast sessions and $n_s \geq \frac{4d\sqrt{c}}{c_6}\sqrt{\frac{n \log n}{k}}$. With probability at least $(1-\frac{2}{n})^2 \geq 1-\frac{4}{n}$, the achievable per-flow multicast capacity is at least*

$$\lambda_k(n) = \frac{W}{3\sqrt{c_6}\Delta} \cdot \frac{a}{n_s r\sqrt{k}} = \Theta(\frac{W}{n_s\sqrt{k}} \cdot \frac{a}{r}). \quad (23)$$

By setting $\frac{a}{r} = \sqrt{\frac{P_{rnd} \cdot n}{\log n}}$ for a constant $c \leq 1/160$, we know that the achievable aggregated multicast capacity is at least $\Theta(\frac{\sqrt{nP_{rnd}}}{\sqrt{\log n}\cdot\sqrt{k}} \cdot W)$, and the per-flow multicast capacity is at least $\Theta(\frac{\sqrt{P_{rnd}}}{\sqrt{\log n}\cdot\sqrt{kn}} \cdot W)$ when there are at least $n_s = \Omega(\sqrt{\frac{n \log n}{k}} \cdot \log k)$ randomly and independently chosen flows.

4.4 Capacity Lower Bound When $k = \Omega(\frac{a^2}{r^2})$

In this case, we have proved that the upper bound on the total multicast capacity is only $\Theta(W)$. Obviously, the total multicast capacity is at least the lower bound of the capacity for broadcast. In [7], they present a broadcast scheme to achieve capacity $\Theta(W)$. Thus, we have the following theorem

THEOREM 27. *The total multicast capacity $\Lambda_k(n)$ achievable by all multicast flows is at least $c_7 W$ when $k = \Omega(a^2/r^2)$, where $c_7 = \frac{1}{\Delta+1}$ and constant Δ is the maximum number of CDS nodes that are within interference range R of a node.*

5. LITERATURE REVIEWS

Network capacity has been extensively studied recently. For a given statistical description of the network.Kyasanur and Vaidya [10] studied the capacity region on **given** multi-hop multi-radio multi-channel wireless networks when there are total c channels available and each node has m wireless interfaces with $m \leq c$. On the other aspect, several papers [1, 9] recently studied how to satisfy a certain traffic demand vector from all wireless nodes by a joint routing, link scheduling, and channel assignment under certain wireless interference models. Gupta and Kumar [5] studied the asymptotic *unicast* capacity of **random** multi-hop wireless networks for two different models. Grossglauser and Tse [4] recently showed that mobility actually can help to improve the unicast capacity if we allow arbitrary large delay. Broadcast capacity of an arbitrary network has been studied in [7, 15]. Multicast capacity was not fully studied in the literature. Jacquet and Rodolakis [6] studied the scaling properties of multicast for random wireless networks. In [12], Li *et al.* also studied the multicast capacity for random networks of singe-channel model. For multi-channel singe-radio networks, Bhandari and Vaidya [3] studied the unicast capacity and connectivity requirement for unicast.

6. CONCLUSIONS

In this paper, we derived matching analytical upper bounds and lower bounds on multicast capacity of a wireless network with multi-channel and multi-radio (MCMR) when all nodes are uniformly and randomly deployed in a square region with side-length a, and all nodes have the same transmission range r. Observe that all our results are proved when the deployment region is a square with side-length a and the transmission ranges of all nodes are uniform with value r. We can show that all our results still apply when the deployment region is a square with side length $a = 1$, while the transmission range is selected appropriately, *i.e.*, $r = \Theta(\sqrt{\frac{\log n}{\pi n}})$. It is surprising that increasing the number of radios cannot improve the asymptotic capacity of a random network.

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