Solutions HW #2

Problem 1:

(1) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

let $\mathbf{a} = a_i \mathbf{e}_i$, $\mathbf{b} = b_j \mathbf{e}_j$, $\mathbf{c} = c_k \mathbf{e}_k$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times (\epsilon_{jkl}\, b_j c_k \mathbf{e}_l)$$

$$= \underbrace{\epsilon_{ilm}\,\epsilon_{jkl}}\, a_i b_j c_k \mathbf{e}_m$$

$$= (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})\, a_i b_j c_k \mathbf{e}_m$$

$$= a_k c_k b_m \mathbf{e}_m - a_i b_i c_m \mathbf{e}_m$$

$$= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

(2) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = a_i\mathbf{e}_i \cdot (\epsilon_{jkl}\, b_j c_k \mathbf{e}_l)$$

$$= a_i b_j c_k\, \epsilon_{jkl}\, \delta_{il} = \underline{a_i b_j c_k\, \epsilon_{jki}}$$

$$\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = b_j\mathbf{e}_j \cdot (\epsilon_{kil}\, c_k a_i \mathbf{e}_l)$$

$$= a_i b_j c_k\, \epsilon_{kil}\, \delta_{jl} = \underline{a_i b_j c_k\, \epsilon_{kij}}$$

since $\epsilon_{jki} = \epsilon_{kij} \Rightarrow \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

Problem 2:

(1) show that $\nabla\cdot(\nabla\times\mathbf{V}) = 0 \leftarrow \nabla = \dfrac{\partial}{\partial x_m}\mathbf{e}_m$

$$\Rightarrow \nabla\cdot(\nabla\times\mathbf{V}) = \frac{\partial}{\partial x_i}\mathbf{e}_i \left(\frac{\partial}{\partial x_j}\mathbf{e}_j \times V_k \mathbf{e}_k\right)$$

$$= \frac{\partial^2 V_k}{\partial x_i \partial x_j}\mathbf{e}_i \cdot \epsilon_{jkl}\mathbf{e}_l = \frac{\partial^2 V_k}{\partial x_i \partial x_j}\overbrace{\epsilon_{jkl}\,\delta_{il}}^{\epsilon_{jki} = \epsilon_{ijk}}$$

$$= \epsilon_{ijk}\frac{\partial^2 V_k}{\partial x_i \partial x_j} = \cancel{\frac{\partial^2 V_3}{\partial x_1 x_2}} + \cancel{\frac{\partial^2 V_1}{\partial x_2 x_3}} + \cancel{\frac{\partial^2 V_2}{\partial x_3 \partial x_1}}$$

$$- \cancel{\frac{\partial^2 V_3}{\partial x_2 x_1}} - \cancel{\frac{\partial^2 V_1}{\partial x_3 \partial x_2}} - \cancel{\frac{\partial^2 V_2}{\partial x_1 \partial x_3}} = 0$$

(2) $\underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\varphi)=\nabla^2\varphi$

$$\underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\varphi)=\frac{\partial}{\partial x_i}\underset{\sim}{e}_i\cdot\left(\frac{\partial\varphi}{\partial x_j}\underset{\sim}{e}_j\right)=\frac{\partial^2\varphi}{\partial x_i x_j}\delta_{ij}$$

$$=\frac{\partial^2\varphi}{\partial x_i\partial x_i}=\underline{\nabla^2\varphi}$$

<u>Problem 3</u>

(1) $M_{ij}\delta_{ij}=M_{ii}=M_{11}+M_{22}+M_{33}$

(2) $\delta_{ii}=\delta_{11}+\delta_{22}+\delta_{33}=3$

(3) $x_i y_j\delta_{ij}-x_q y_p\delta_{qp}=x_i y_i-x_q y_q=0$

(4) makes no sense! Left side of eqn. has 3 free indices while right hand side has only one free index, it is not compatible.

(5) $(\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ij})=(3\times 3-\underbrace{\delta_{ii}}_{3})=6$

<u>Problem 4:</u>

(1) solve the characteristic eqn. to find eigenvalues $(\lambda_1>\lambda_2>\lambda_3)$

$\det(\underset{\approx}{A}-\lambda\underset{\approx}{I})=0$

$$\begin{vmatrix}1-\lambda & 2 & 3\\ 2 & 4-\lambda & 6\\ 3 & 6 & 1-\lambda\end{vmatrix}=0=\lambda(\lambda+4)(\lambda-10)$$

$$\Rightarrow\boxed{\lambda_1=10},\ \boxed{\lambda_2=0},\ \boxed{\lambda_3=-4}$$

(2) find corresponding principal directions $\underset{\sim}{\nu}^1,\underset{\sim}{\nu}^2,\underset{\sim}{\nu}^3$

for $\lambda_1=10$ $\qquad\underset{\approx}{A}\underset{\sim}{\nu}^{(i)}=\lambda_i\underset{\sim}{\nu}^{(i)}$ (no sum)

$$\begin{bmatrix}1-10 & 2 & 3\\ 2 & 4-10 & 6\\ 3 & 6 & 1-10\end{bmatrix}\begin{bmatrix}\nu_1^{(1)}\\ \nu_2^{(1)}\\ \nu_3^{(1)}\end{bmatrix}=0$$

$$\begin{bmatrix} -9 & 2 & 3 \\ 2 & -6 & 6 \\ 3 & 6 & -9 \end{bmatrix} \to \begin{bmatrix} -9 & 2 & 3 \\ 20 & -10 & 0 \\ -24 & 12 & 0 \end{bmatrix} \to \begin{bmatrix} -9 & 2 & 3 \\ 20 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -9v_1 + 2v_2 + 3v_3 = 0 \\ 20v_1 - 10v_2 = 0 \end{array}\right\}$$ there are only 2 independent eqns for 3 unknowns (v_1, v_2, v_3)

Choose $v_2 = 1$, then $v_1 = 1/2$

$-9 \cdot \frac{1}{2} + 2 \cdot 1 + 3v_3 = 0 \implies v_3 = 5/6$

$$\underset{\sim}{v}^{(1)} = \begin{bmatrix} 1/2 \\ 1 \\ 5/6 \end{bmatrix} \xrightarrow[\text{normalized}]{/\frac{\sqrt{70}}{6}} \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} \cdot \frac{1}{\sqrt{70}}$$

for $\lambda_2 = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \left.\begin{array}{l} z = 0 \\ x = -2y \end{array}\right\}$$ choose $y = 1$, $x = -2$

$$\underset{\sim}{v}^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \xrightarrow[\text{normalized}]{/\sqrt{5}} \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda_3 = -4$

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 8 & 6 \\ 3 & 6 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 2 & 3 \\ 0 & 36/5 & 24/5 \\ 0 & 24/5 & 16/5 \end{bmatrix} \to \begin{bmatrix} 5 & 2 & 3 \\ 0 & \frac{36}{5} & \frac{24}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

choose $y = 1 \Rightarrow z = -\frac{3}{2} \Rightarrow x = \frac{1}{2}$

$$\Rightarrow \underset{\sim}{\nu}^{(3)} = \begin{bmatrix} 1/2 \\ 1 \\ -3/2 \end{bmatrix} \xrightarrow[\text{normalized}]{/\sqrt{14}/2} \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

(3)

$$\left.\begin{aligned} \underset{\sim}{\nu}^{(1)} \cdot \underset{\sim}{\nu}^{(2)} &= \begin{bmatrix} 1/2 & 1 & 5/6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = 0 \\ \underset{\sim}{\nu}^{(2)} \cdot \underset{\sim}{\nu}^{(3)} &= \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ -3/2 \end{bmatrix} = 0 \\ \underset{\sim}{\nu}^{(1)} \cdot \underset{\sim}{\nu}^{(3)} &= \begin{bmatrix} 1/2 & 1 & 5/6 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ -3/2 \end{bmatrix} = 0 \end{aligned}\right\} \Rightarrow$$

$\underset{\sim}{\nu}^{(1)}, \underset{\sim}{\nu}^{(2)}, \underset{\sim}{\nu}^{(3)}$ are mutually orthogonal!

(4) find the invariants in terms of d_1, d_2, d_3

$$[\underset{\approx}{A}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \xrightarrow[\text{(spectral form)}]{\text{diagonalized}} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\left.\begin{aligned} I_1 &= tr(\underset{\approx}{A}) = d_1 + d_2 + d_3 = 10 + 0 - 4 = \underline{\underline{6}} \\ I_2 &= \begin{vmatrix} 10 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & -4 \end{vmatrix} + \begin{vmatrix} 10 & 0 \\ 0 & -4 \end{vmatrix} = \underline{\underline{-40}} \\ I_3 &= det(\underset{\approx}{A}) = d_1 d_2 d_3 = (10)(0)(-4) = \underline{\underline{0}} \end{aligned}\right\}$$

in fact we can also calculate the invariants $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ cartesian frame

$$\left\{\begin{aligned} I_1 &= tr(\underset{\approx}{A}) = 1 + 4 + 1 = \underline{\underline{6}} \\ I_2 &= \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 6 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 0 + (-32) + (-8) \\ &= \underline{\underline{-40}} \\ I_3 &= det(\underset{\approx}{A}) = 1 \cdot (-32) - 2(-16) + 3(0) = \underline{\underline{0}} \end{aligned}\right.$$

The results (I_1, I_2, I_3) are the same in principal frame and initial cartesian frame. Actually, those 3 values hold the same in all frames That's why they are called "invariants"