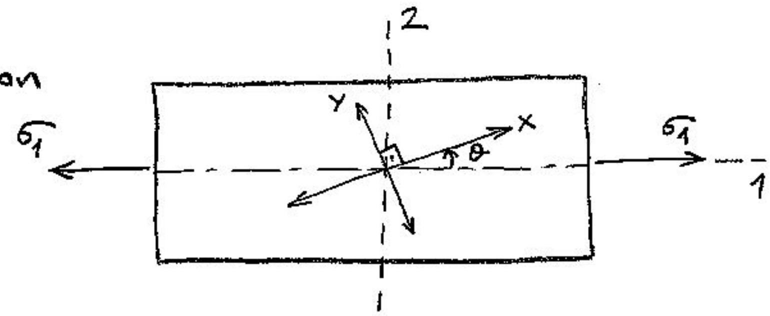
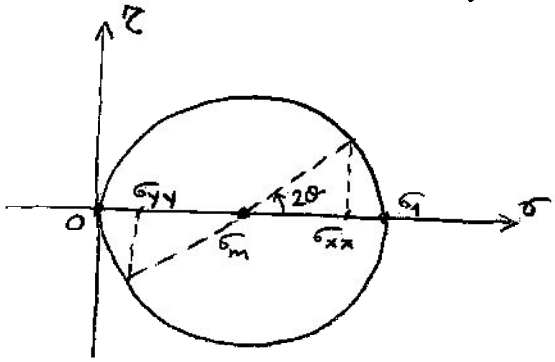
SOLUTIONS HW # 5

Problem 1

In a uniaxial tension/compression experiment we are measuring ε_{xx} and ε_{yy} as shown in the sketch ($\theta \rightarrow$ unknown).



We can quickly analyze this situation using Mohr's circle for uniaxial loading



mean stress: $\sigma_m = \dfrac{\sigma_{xx}+\sigma_{yy}}{2}$

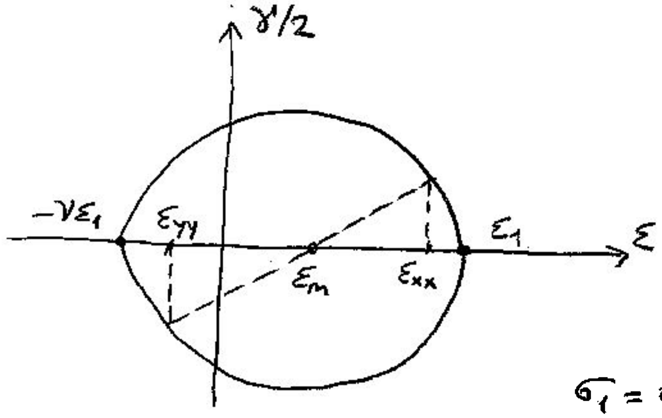
radius: $R = \sigma_m = \dfrac{\sigma_{xx}+\sigma_{yy}}{2}$

$$\Rightarrow \sigma_1 = \sigma_m + R = \sigma_{xx}+\sigma_{yy} \quad \text{---- (1)}$$

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy}) \\ \varepsilon_{yy} &= \frac{1}{E}(\sigma_{yy}-\nu\sigma_{xx}) \end{aligned}\right\} \Rightarrow \varepsilon_{xx}+\varepsilon_{yy} = \frac{1}{E}(1-\nu)(\sigma_{xx}+\sigma_{yy}) \quad \text{------ (2)}$$

Combining (1) & (2) $\longrightarrow$ $\boxed{\sigma_1 = \dfrac{E}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy})}$

* ALTERNATIVELY, this problem can be solved using Mohr's circle for strain:



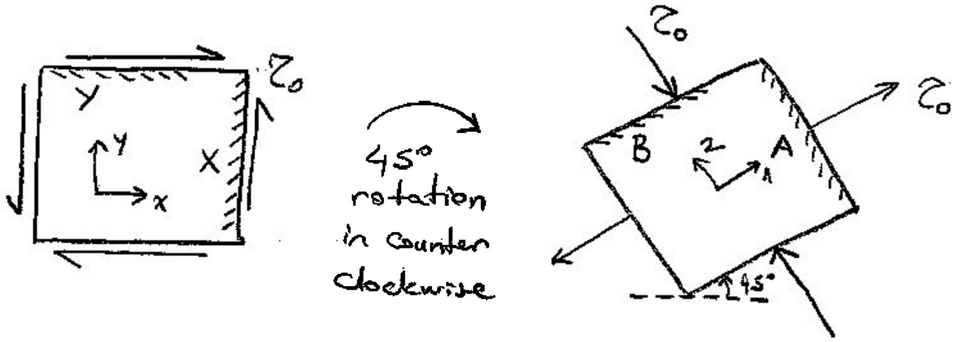
$$\varepsilon_m = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{\cancel{2}} = \frac{\varepsilon_1 - \nu\varepsilon_1}{\cancel{2}}$$

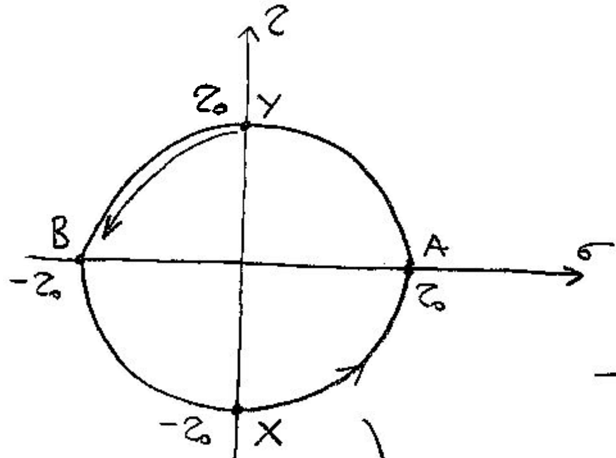
$$\Rightarrow \varepsilon_1 = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{1-\nu}$$

$\sigma_1 = E \cdot \varepsilon_1$ for uniaxial loading

$$\Rightarrow \boxed{\sigma_1 = \frac{E}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy})}$$

— Let's consider a pure shear loading and 45° transformation:

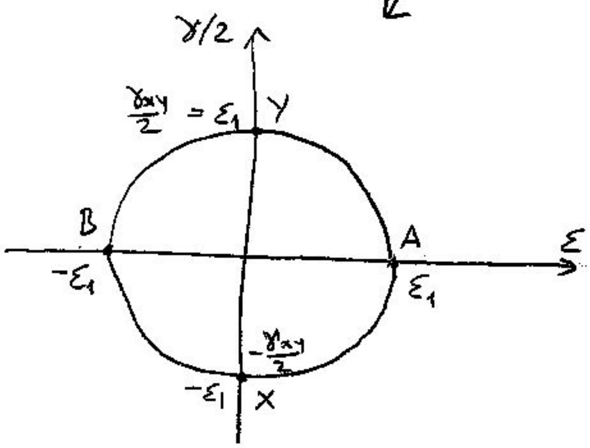




Mohr's circle corresponding to a pure shear loading is given in the sketch. After 45° C-Cwise rotation surface X transforms to A, and Y to B.

— We can write in the transformed state:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}(\tau_0 + \nu \tau_0) \\ \varepsilon_2 &= \frac{1}{E}(-\tau_0 - \nu \tau_0) \end{aligned}\right\} \Rightarrow \varepsilon_1 = \frac{(1+\nu)}{E}\tau_0 = -\varepsilon_2$$



— On the other hand, in original state:

$$\gamma_{xy} = \frac{1}{\mu}\tau_0$$

it's apparent from strain Mohr's circle that $\frac{\gamma_{xy}}{2} = \varepsilon_1$

$$\Rightarrow 2\varepsilon_1 = \frac{1}{\mu}\tau_0$$

$$\frac{2(1+\nu)}{E}\cancel{\tau_0} = \frac{1}{\mu}\cancel{\tau_0} \Rightarrow \boxed{\mu = \frac{E}{2(1+\nu)}}$$

Problem 3 For an isotropic linear elastic solid

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

a) $\sigma_{11} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{11}$ $\quad$ $\sigma_{12} = 2\mu\varepsilon_{12}$

$\sigma_{22} = \lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{22}$ $\quad$ $\sigma_{13} = 2\mu\varepsilon_{13}$

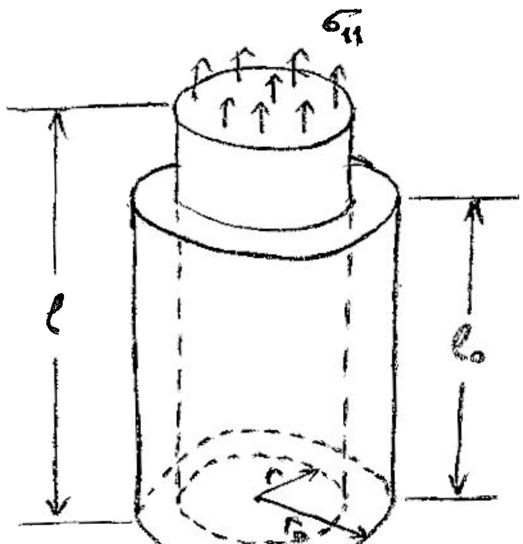
noting that the strain (ε_{ij}) is a unitless quantity, the Lame's constants (λ, μ) have the unit of stress [Force/Area] → MPa, msi, etc.

b) For the uniaxial stress state, we have

$$\left.\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\sigma_{11} \\ \varepsilon_{22} &= \frac{1}{E}(-\nu\sigma_{11}) = -\nu\varepsilon_{11} \\ \varepsilon_{33} &= \frac{1}{E}(-\nu\sigma_{11}) = -\nu\varepsilon_{11} \end{aligned}\right\}$$

the unit of E is $\frac{N}{m^2} = Pa$,

ν is unitless (non-dimensional)



If the undeformed radius and length are r_0 and l_0, and the radius and length after deformation are r and l, then

$$\varepsilon_{11} = \frac{l-l_0}{l_0} \qquad \varepsilon_{22} = \frac{r-r_0}{r_0}$$

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{r_0-r}{r_0}\ \frac{l_0}{l-l_0}$$

Problem 4 For an isotropic linear elastic solid

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \text{ --------- } (1)$$

$$\varepsilon_{kk} = \varepsilon_{kl} \delta_{kl} \text{ --------- } (2)$$

$$\varepsilon_{ij} = \frac{1}{2} (\varepsilon_{ij} + \varepsilon_{ji})$$ ← to preserve the symmetry of ε_{ij}

$$= \frac{1}{2} (\varepsilon_{kl} \delta_{ik} \delta_{jl} + \varepsilon_{kl} \delta_{jk} \delta_{il}) \text{ ----- } (3)$$

substituting (2) & (3) into (1), we get

$$\sigma_{ij} = \underbrace{\left[\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})\right]}_{C_{ijkl}} \varepsilon_{kl}$$

Problem 5

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \quad \text{-----} \quad (1)$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \quad \text{----} \quad (2)$$

from constitutive relation

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad \text{-----} \quad (3)$$

$$\Rightarrow \quad \sigma_{kk} = \lambda\varepsilon_{kk}\cdot 3 + 2\mu\varepsilon_{kk}$$

$$\Rightarrow \quad \underbrace{\frac{\sigma_{kk}}{3}}_{-p} = \underbrace{(\lambda + \tfrac{2}{3}\mu)}_{K:\text{ bulk modulus}}\varepsilon_{kk} \quad \text{-------} \quad (4)$$

substituting (3) & (4) in (1) gives :

$$S_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (\lambda + \tfrac{2}{3}\mu)$$

$$= 2\mu\underbrace{(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij})}_{e_{ij}} \Rightarrow \boxed{S_{ij} = 2\mu e_{ij}}$$

And from (4) $\longrightarrow$ $\boxed{p = -K\varepsilon_{kk}}$

This analysis shows that any stress state can be decomposed into hydrostatic (p) and deviatoric (S_{ij}) parts, and hydrostatic part is responsible for volume change (ε_{kk}) in the body while the deviatoric part generates shape change (e_{ij}) without affecting on volume.