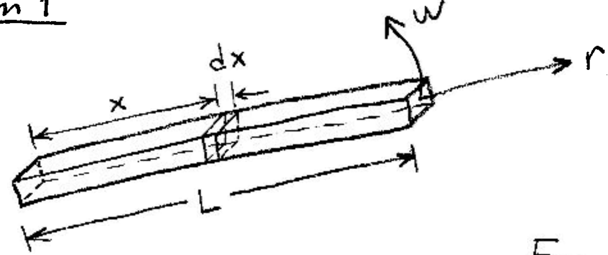
MIDTERM SOLUTIONS

Problem 1



angular velocity $w = 2\pi N$

Equilibrium eqn:

$$\sigma_{ij,j} + b_i = \rho a_i$$

For 1-D problem

$$\frac{\partial \sigma_{xx}}{\partial x} + \cancel{b_x}^{\,0} = \rho \underbrace{a_x}_{-w^2 x}$$

angular acceleration is directed towards the axis of rotation!

$$\Rightarrow \sigma_{xx} = \int -\rho w^2 x \, dx$$

$$\sigma_{xx} = -\frac{1}{2}\rho w^2 x^2 + C_0$$

B.C. $\sigma_{xx} = 0$ at $x = L$ (traction free edge)

$$0 = -\frac{1}{2}\rho w^2 L^2 + C_0 \Rightarrow C_0 = \frac{1}{2}\rho w^2 L^2$$

$$\Rightarrow \boxed{\sigma_{xx} = \frac{1}{2}\rho w^2 (L^2 - x^2)}$$

(a) Obviously max. stress occurs at $x = \emptyset$

$$\sigma_{max} = \frac{1}{2}\rho w^2 L^2 = 2\rho \pi^2 N^2 L^2$$

(b) $\left.\begin{array}{l} \sigma = E \cdot \varepsilon \\ \varepsilon = \dfrac{du}{dx} \end{array}\right\}$ $\dfrac{du}{dx} = \dfrac{\sigma}{E} = \dfrac{\rho w^2}{2E}(L^2 - x^2)$

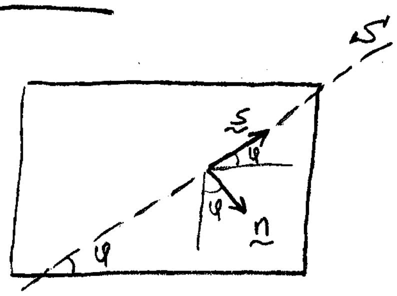
$$u = \frac{\rho w^2}{2E}\left(L^2 x - \frac{x^3}{3}\right) + C_1$$

B.C.: $u = 0$ at $x = 0 \Rightarrow C_1 = \emptyset$

max. displacement occurs at $x = L$, where $\left.\dfrac{du}{dx}\right|_{x=L} = \emptyset$

$$U_{max} = u\Big|_{x=L} = \frac{\rho w^2}{2E}\left(L^3 - \frac{L^3}{3}\right) = \frac{\rho w^2 L^2}{3E} = \frac{4}{3}\rho \pi^2 N^2 L^3$$

Problem 2



$$\underset{\sim}{n} = \begin{Bmatrix} \sin\varphi \\ -\cos\varphi \end{Bmatrix} = \begin{Bmatrix} s \\ -c \end{Bmatrix} \quad \text{and} \quad \underset{\sim}{s} = \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix} = \begin{Bmatrix} c \\ s \end{Bmatrix}$$

surface normal ← UNIT VECTORS → surface tangent

stress field $\underset{\approx}{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

(a) traction acting on surface $\mathcal{S}$

$$\underset{\sim}{t}(\underset{\sim}{n}) = \underset{\approx}{\sigma} \cdot \underset{\sim}{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} s \\ -c \end{bmatrix} = \begin{bmatrix} s\sigma_{11} - c\sigma_{12} \\ s\sigma_{12} - c\sigma_{22} \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

(b) normal stress

$$\sigma_n = \underset{\sim}{t} \cdot \underset{\sim}{n} = s^2\sigma_{11} + c^2\sigma_{22} - 2cs\,\sigma_{12}$$

shear stress

$$\left.\begin{aligned} \sigma_t = \underset{\sim}{t} \cdot \underset{\sim}{s} &= cs\,\sigma_{11} - c^2\sigma_{12} + s^2\sigma_{12} - cs\,\sigma_{22} \\ &= (\sigma_{11} - \sigma_{22})\,cs - (c^2 - s^2)\,\sigma_{12} \\ &= \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\sin 2\varphi - \sigma_{12}\cos 2\varphi \end{aligned}\right\} \leftarrow \begin{aligned} c^2 - s^2 &= \cos 2\varphi \\ cs &= \tfrac{1}{2}\sin 2\varphi \end{aligned}$$

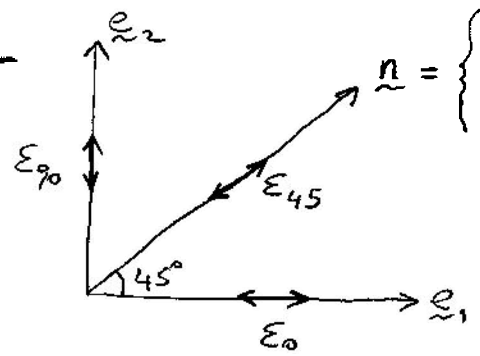
(c) if the plane is traction free $\Rightarrow$ $\underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\Rightarrow \left.\begin{aligned} s\sigma_{11} - c\sigma_{12} &= 0 \\ s\sigma_{12} - c\sigma_{22} &= 0 \end{aligned}\right\} \begin{aligned} &\longrightarrow \sigma_{12} = \frac{s}{c}\sigma_{11} \\ &\longrightarrow \frac{s^2}{c}\sigma_{11} - c\sigma_{22} = 0 \longrightarrow \sigma_{22} = \frac{s^2}{c^2}\sigma_{11} \end{aligned}$$

$$\boxed{\sigma_{22} = \tan^2\varphi\,\sigma_{11}}$$

condition for surface $\mathcal{S}$ to be traction free

Problem 3



$\underset{\sim}{n} = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}$ ← normalized!

③

strain tensor $\underset{\approx}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix}$

(a) components of strain tensor ($\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$):

$$\varepsilon_0 = \underset{\sim}{e}_1 \cdot \underset{\approx}{\varepsilon} \cdot \underset{\sim}{e}_1 = \varepsilon_{11}$$

$$\varepsilon_{90} = \underset{\sim}{e}_2 \cdot \underset{\approx}{\varepsilon} \cdot \underset{\sim}{e}_2 = \varepsilon_{22}$$

$$\varepsilon_{45} = \underset{\sim}{n} \cdot \underset{\approx}{\varepsilon} \cdot \underset{\sim}{n} = \frac{1}{2}\left(\varepsilon_{11} + \varepsilon_{22} + 2\varepsilon_{12}\right) \Rightarrow \boxed{\varepsilon_{12} = \varepsilon_{45} - \left(\frac{\varepsilon_0 + \varepsilon_{90}}{2}\right)}$$

(b) principal strains

$$\begin{vmatrix} \varepsilon_{11} - \lambda & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} - \lambda \end{vmatrix} = 0 = \lambda^2 - \lambda(\varepsilon_{11} + \varepsilon_{22}) + \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2$$

$$\lambda_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}, \quad \Delta = (\varepsilon_{11} + \varepsilon_{22})^2 - 4(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)$$

$$= \varepsilon_{11}^2 + \varepsilon_{22}^2 - 2\varepsilon_{11}\varepsilon_{22} + 4\varepsilon_{12}^2$$

$$= (\varepsilon_{11} - \varepsilon_{22})^2 + 4\varepsilon_{12}^2$$

$$\Rightarrow \lambda_{1,2} = \left(\frac{\varepsilon_{11} + \varepsilon_{22}}{2}\right) \mp \frac{1}{2}\sqrt{(\varepsilon_{11} - \varepsilon_{22})^2 + 4\varepsilon_{12}^2}$$

$$\boxed{\lambda_{1,2} = \frac{\varepsilon_{11} + \varepsilon_{22}}{2} \mp \sqrt{\left(\frac{\varepsilon_{11} - \varepsilon_{22}}{2}\right)^2 + \varepsilon_{12}^2}}$$ principal strains

(c) for $\varepsilon_{45} = \left(\frac{\varepsilon_0 + \varepsilon_{90}}{2}\right)$

$$\left.\begin{aligned} \varepsilon_{11} &= \varepsilon_0 \\ \varepsilon_{22} &= \varepsilon_{90} \\ \varepsilon_{12} &= \left(\frac{\varepsilon_0 + \varepsilon_{90}}{2}\right) - \left(\frac{\varepsilon_0 + \varepsilon_{90}}{2}\right) = \emptyset \end{aligned}\right\}$$

since shear strain $\varepsilon_{12} = 0$, $e_1 - e_2$ frame is the principal frame!

And principal strains are

$\varepsilon_1 = \varepsilon_0$, $\varepsilon_2 = \varepsilon_{90}$