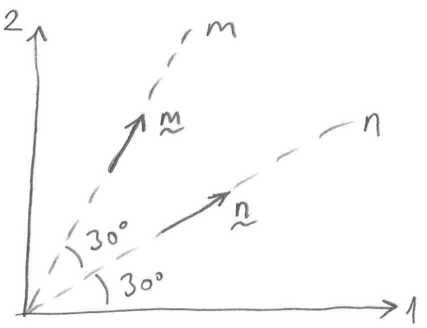
MMAE 530 MAKEUP EXAM SOLUTIONS

(1)



$$\underset{\sim}{n} = \begin{Bmatrix} \cos 30° \\ \sin 30° \end{Bmatrix} = \begin{Bmatrix} c \\ s \end{Bmatrix}$$

where
$c = \cos 30° = \frac{\sqrt{3}}{2}$
$s = \sin 30° = \frac{1}{2}$

$$\underset{\sim}{m} = \begin{Bmatrix} \cos 60° \\ \sin 60° \end{Bmatrix} = \begin{Bmatrix} s \\ c \end{Bmatrix}$$

$\varepsilon_0 = \varepsilon_{11} = 10^{-3}$; $\varepsilon_{22} = ?$; $\varepsilon_{12} = ?$

$$\varepsilon_{30} = \underset{\sim}{n} \cdot \underset{\approx}{\varepsilon} \cdot \underset{\sim}{n} = \{c \quad s\} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} \begin{Bmatrix} c \\ s \end{Bmatrix} = c^2 \varepsilon_{11} + s^2 \varepsilon_{22} + 2cs\,\varepsilon_{12} \quad \text{----- (1)}$$

$$\varepsilon_{60} = \underset{\sim}{m} \cdot \underset{\approx}{\varepsilon} \cdot \underset{\sim}{m} = \{s \quad c\} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} \begin{Bmatrix} s \\ c \end{Bmatrix} = s^2 \varepsilon_{11} + c^2 \varepsilon_{22} + 2cs\,\varepsilon_{12} \quad \text{----- (2)}$$

(1)−(2) gives $\Longrightarrow$ $\varepsilon_{22} = \dfrac{\varepsilon_{30} - \varepsilon_{60} + (s^2 - c^2)\,\varepsilon_0}{(s^2 - c^2)}$ $\longleftarrow$ $(s^2 - c^2) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$

$$\boxed{\varepsilon_{22} = \varepsilon_0 + 2(\varepsilon_{60} - \varepsilon_{30})}^{*} = \{1 + 2(3-2)\} \times 10^{-3} = \underline{\underline{3 \times 10^{-3}}}$$

(1)+(2) gives $\Longrightarrow$ $\varepsilon_{12} = \dfrac{\varepsilon_{30} + \varepsilon_{60} - \varepsilon_0 - \varepsilon_{22}}{4cs}$ $\longleftarrow$ $4cs = \sqrt{3}$ & ε_{22}

$$\boxed{\varepsilon_{12} = \frac{3\varepsilon_{30} - \varepsilon_{60} - 2\varepsilon_0}{\sqrt{3}}}^{*} = \left\{ \frac{3 \times 2 - 3 - 2 \times 1}{\sqrt{3}} \right\} \times 10^{-3} = \underline{\underline{\frac{1}{\sqrt{3}} \times 10^{-3}}}$$

(2) (a) prove that $\underbrace{\underset{\sim}{\mu} \cdot \underset{\sim}{t}^{(\nu)}} = \underbrace{\underset{\sim}{\nu} \cdot \underset{\sim}{t}^{(\mu)}}$ $\longleftarrow$ $\underset{\sim}{t}^{(\nu)} = \underset{\approx}{\sigma} \cdot \underset{\sim}{\nu}$ AND $\underset{\sim}{t}^{(\mu)} = \underset{\approx}{\sigma} \cdot \underset{\sim}{\mu}$

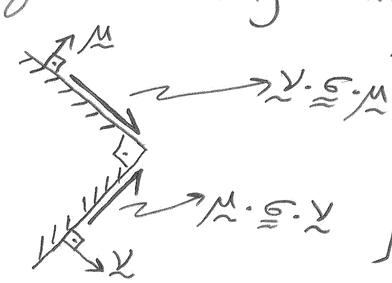
$$\underset{\sim}{\mu} \cdot \underset{\approx}{\sigma} \cdot \underset{\sim}{\nu} = \underset{\sim}{\nu} \cdot \underset{\approx}{\sigma} \cdot \underset{\sim}{\mu}$$

$$\mu_i \sigma_{ij} \nu_j = \nu_i \sigma_{ij} \mu_j$$

$$= \nu_j \sigma_{ji} \mu_i \longleftarrow (\text{since } \sigma_{ij} = \sigma_{ji})$$

$$= \mu_i \sigma_{ij} \nu_j \quad \checkmark \quad (\text{proved by using the symmetry of } \sigma_{ij})$$

(b) this relation takes a familiar meaning when we consider two orthogonal planes (i.e., $\underset{\sim}{\mu} \perp \underset{\sim}{\nu}$)



as seen from the sketch this relation implies the equality of shear tractions on perpendicular plane!

③ $x_1' = x_1\cos\theta + x_2\sin\theta$

$x_2' = -x_1\sin\theta + x_2\cos\theta$

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}'}{\partial \underset{\sim}{x}} \quad \text{OR} \quad F_{ij} = \frac{\partial x_i'}{\partial x_j}$$

(a)

$$(\underset{\sim}{F}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underset{\approx}{E} = \frac{1}{2}(\underset{\approx}{F}^T\underset{\approx}{F} - \underset{\approx}{I})$$

$$\underset{\approx}{E} = \frac{1}{2}\left[\begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ NO deformation

Lagrangian strain tensor gives us zero-deformation, as it should be, because this is a rigid body rotation!

(b) $\underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{u} \longrightarrow \underset{\sim}{u} = \underset{\sim}{x}' - \underset{\sim}{x}$

$\Rightarrow \quad u_1 = x_1' - x_1 = x_1(\cos\theta - 1) + x_2\sin\theta$

$u_2 = x_2' - x_2 = -x_1\sin\theta + x_2(\cos\theta - 1)$

$u_3 = x_3' - x_3 = \emptyset$

displacement gradient:

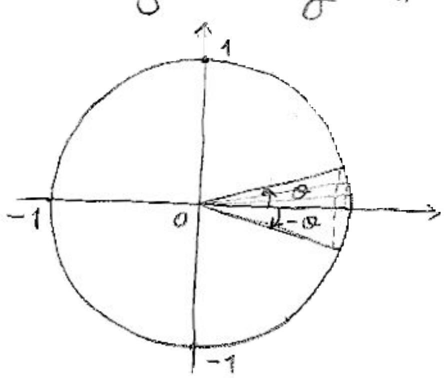
$$\underset{\sim}{\nabla}\underset{\sim}{u} = \frac{\partial u_i}{\partial x_j}$$

$$\underset{\sim}{\nabla}\underset{\sim}{u} = \begin{pmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

small strain tensor $\Big\}$ $$\underset{\approx}{\varepsilon} = \frac{1}{2}(\underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{u}^T) = \begin{pmatrix} \cos\theta - 1 & 0 & 0 \\ 0 & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ NON-ZERO strains!

(c) extensional strain in $\underset{\sim}{e}_1$ direction $= \varepsilon_{11} = \underset{\sim}{e}_1 \cdot \underset{\approx}{\varepsilon} \cdot \underset{\sim}{e}_1 = \cos\theta - 1$

magnitude of $\varepsilon_{11} \longrightarrow |\varepsilon_{11}| = |\cos\theta - 1| < 0.01$



$-0.01 < \cos\theta - 1 < 0.01$

$0.99 < \cos\theta < 1.01$

$0.99 < \cos\theta \leq 1$

$$\boxed{-8.1^\circ < \theta < 8.1^\circ}$$

(d) In a rigid body rotation, Lagrangian strains are correctly zero while infinitesimal strain components are not zero This is because we omitted 2nd order terms in the definition of $\underset{\approx}{\varepsilon}$ and therefore $\underset{\approx}{\varepsilon}$ is not an exact measure of strains particularly for large rigid body rotations

Recall that Lagrangian strain

$$\underset{\approx}{E} = \frac{1}{2}\left(\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T + \underbrace{\nabla \underset{\sim}{u} (\nabla \underset{\sim}{u})^T}_{} \right)$$

this quadratic term gets larger because large rotations bring large displacement gradients even without any real deformation!