1. Let \( l^\infty \) denote all bounded sequences of real numbers. For \( x = (x_1, x_2, \ldots) \in l^\infty \), define
\[
\|x\| = \sup_i |x_i|.
\] (1)

(a) Show that \( l^\infty \) equipped with Eq. (1) is a normed linear space.
(b) Is \( l^\infty \) a Banach space? Why?

2. For \( x_n = \ln(n+2) - \ln n \) for \( n = 1, 2, \ldots \), find \( \lim \sup x_n \) and \( \lim \inf x_n \).

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1.3 (a) Let \( x = (x_1, x_2, \ldots, x_n, \ldots) \), \( y = (y_1, y_2, \ldots, y_n, \ldots) \).
\[
x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots), \quad \lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots)
\]
Since the addition and scalar multiplication are defined componentwise, the properties of real numbers lead to the properties of linear space:
\[
x + y = y + x, \quad (x + y) + z = x + (y + z), \quad \lambda(x + y) = \lambda x + \lambda y,
\]
\[
\|\lambda x\| = |\lambda| \|x\|.
\]
Also the sum of bounded seq is bounded, that the scalar multipli- of
a bdd seq is bdd.

\[\varepsilon \]

\[
\text{norm} : \quad \|x\| = \sup_i |x_i| \geq 0 \quad (\text{non-neg.})\]
\[
\|\lambda x\| = |\lambda| \|x\|.\]

If \( \|x\| = 0 \), then \( x = 0 \) for all \( i \). \( \sup_i |x_i| = 0 \).

Thus, \( \|x\| \) is a norm.

(b) \( l^\infty \) is a Banach space.

**Pf:** Let \( \{x^{(k)}\}_{k=1}^{\infty} \) be a Cauchy seq in \( l^\infty \). Then, \( \{x^{(k)}\}_{k=1}^{\infty} \) is a Cauchy seq. in \( \mathbb{R} \) for each \( i \).

\( \mathbb{R} \) is complete. \( (x^{(k)}) \) converges as \( k \to \infty \), denote the limit by \( x_i \), and \( x = (x_1, x_2, \ldots, x_n, \ldots) \).

Then, \( x^{(n)} \to x \) as \( k \to \infty \). \( x \) is bounded; because \( \{x^{(k)}\} \subset l^\infty \) and is Cauchy seq.

Choose \( \varepsilon = 1 \), \( \|x^{(n)} - x^{(m)}\| \leq 1 \) for some \( N_n > 0 \) and \( m > N \).

Thus, \( \|x^{(n)}\| = \|x^{(n)} + x^{(m)} - x^{(m)}\| \leq \|x^{(m)}\| + \|x^{(m)} - x^{(n)}\| \leq \|x^{(n)}\| + 1 \).

Thus, \( x \in l^\infty \). \( \square \)

2. \( \{x^{(n)}\} \) Cauchy: \( \forall \varepsilon > 0, \exists N_0 \) st. \( \|x^{(m)} - x^{(n)}\| \leq \varepsilon \) for \( m, n > N \).

Let \( n \to \infty \), \( \|x^{(n)} - x^{(n)}\| = \varepsilon \) for \( m > N \). \( \varepsilon \) for \( m > N \). Thus, \( \|x^{(n)} - x\| \leq \varepsilon \) for \( m > N \).

Thus, \( x_n \) is a decreasing seq. of \( n \).

\[
\lim \sup x_n = \lim \inf x_n = \ln 1 = 0
\]

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2. \( x_n = \ln(n+2) - \ln n = \ln n^2/n = \ln(n+2/n) \). \( \ln x \) is an increasing func. of \( x \).

\( x_n \) is a decreasing seq. of \( n \). \( \lim \sup x_n = \lim \inf x_n = \ln 1 = 0 \).