Let $X$ be an uncountable set and $\mathcal{A}$ be the collection of subsets $A$ of $X$ such that either $A$ or $A^c$ is countable.

1. Show that $\mathcal{A}$ is a $\sigma$-algebra.

2. Define $\mu(A) = 0$ if $A$ is countable (including empty set); $\mu(A) = 1$ if $A$ is uncountable. Prove $\mu$ is a measure.

1. (i) $\emptyset \in \mathcal{A}$, b/c $\emptyset$ is countable; $X \in \mathcal{A}$, b/c $X^c = \emptyset$ is countable.

(ii) If $A \in \mathcal{A}$, then $A$ or $A^c$ is countable. Thus, $(A^c)^c = A$, thus $A^c \in \mathcal{A}$.

(iii) If $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots$, then, for each $i$, $A_i$ or $A_i^c$ is countable.

Case 1: if all $A_i$'s are countable, then $\bigcap_{i=1}^\infty A_i$ is countable, $\Rightarrow \bigcap_{i=1}^\infty A_i \in \mathcal{A}$

Case 2: if there $\exists A_j$ is uncountable for some $j$, then $A_j^c$ is countable

and $\left( \bigcup_{i=1}^\infty A_i \right)^c = \bigcap_{i=1}^\infty A_i^c \subseteq A_j^c$, thus $\left( \bigcup_{i=1}^\infty A_i \right)^c$ is countable.

$\Rightarrow \bigcup_{i=1}^\infty A_i \in \mathcal{A}$.

From (i) - (iii) $\mathcal{A}$ is a $\sigma$-algebra.

2. (i) $\emptyset$ is countable $\Rightarrow \mu(\emptyset) = 0$.

(ii) Let $A_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

Case 1: if all $A_i$'s are countable, then $\bigcup_{i=1}^\infty A_i$ is countable,

thus $\mu(\bigcup_{i=1}^\infty A_i) = 0$ and $\sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty 0 = 0 \Rightarrow \mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$

Case 2: if $\exists A_j$ is uncountable for some $j$, then $\bigcup_{i=1}^\infty A_i \setminus A_j \subseteq A_j^c$ is countable.

thus $\mu(\bigcup_{i=1}^\infty A_i) = 1$, $\sum_{i=1}^\infty \mu(A_i) = \mu(A_j) = 0$.

Thus, $\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

Based on (i), (ii), $\mu$ is a measure.